Structures of Digital Filters – II
Topics of last lecture

• Direct-form realizations of digital filters.
• The system (transfer) function.
• Zeros, poles, and the stability.
• The cascade connection.
• The parallel connection.
Topics of this lecture

- State-space representation of digital filters.
- The equivalent transformation.
- Cascade realization.
- Parallel realization.
- Minimum norm form realization.

- デジタルフィルタの状態空間表現
- 等価変換
- 縦続接続
- 並行接続
- 最小ノルム形実現
External representations

- External representations of digital filters
  - Impulse response $h(n)$.
  - Frequency response $H(e^{j\omega})$.
  - Transfer function $H(z)$.
  - Difference equation.

Only the relation between the input and the output is used to analyze the property of the system.
Internal representation  内部表現

• The state-space representation is used to represent the internal structure of a filter.
• For the same external representation, we can have different ways to realize the system.

内部表現は、システムを実装するために必要不可欠である。
Internal representation 内部表現

• For the direct-form II realization, we may define the “states” of the filter as follows (taken from the text):

Each state represents the status of a “probe-point” or “check point”.

\[ s_2(n+1) \]
\[ s_2(n) \]
\[ s_1(n) \]
State transition of the filter

\[
\begin{align*}
    s_1(n+1) &= s_2(n) \\
    s_2(n+1) &= s_3(n) \\
    &\ldots\ldots \\
    s_{N-1}(n+1) &= s_N(n) \\
    s_N(n+1) &= x(n) - a_1 s_N(n) - a_2 s_{N-1}(n) - \ldots - a_N s_1(n)
\end{align*}
\]

内状態が時間によって変わる様子。
微分方程式と似ている。

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The state equation (状态方程式)

- $s_i(n)$ (i=1,2,…,N) are state variables (状態変数).
- $s(n)$ is an N-dimensional state vector.
- The above equation is called the state equation.
- The equation can be written in matrix-vector form

$$s(n+1) = As(n) + Bx(n)$$

- A is called the companion matrix.
- Note here $s(n)$ is the state vector, and $x(n)$ is the input signal from outside.
The output equation (出力方程式)

- From the direct form realization II, we can also get

\[ y(n) = b_N s_1(n) + \cdots + b_2 s_{N-1}(n) + b_1 s_N(n) + b_0 (x(n) - a_N s_1(n) - \cdots - a_2 s_{N-1}(n) - a_1 s_N(n)) \]

\[ = b_0 x(n) + c_1 x_1(n) + \cdots + c_N x_N(n) \]

where

\[ c_i = b_{N-i+1} - b_0 a_{N-i+1}, \quad i = 1, 2, \ldots, N \]
The output equation (出力方程式)

\[ y(n) = [c_1 \ c_2 \ \cdots \ c_N] \begin{bmatrix} s_1(n) \\ s_2(n) \\ \vdots \\ s_N(n) \end{bmatrix} + [b_0]x(n) \]

or in matrix form

\[ y(n) = C s(n) + D x(n) \]
$DF(A,B,C,D)$

- The state equation and the output equation used together is called the state-space representation (状態空間表現) of a digital filter.
- A digital filter can be denoted by $DF(A,B,C,D)$. 
Find the impulse response

- Let $x(n) = \delta(n)$, we can find the impulse response of the filter

\[
 h(n) = \begin{cases} 
 D & n = 0 \\
 CA^{n-1}B & n > 0 
\end{cases} 
\]
Find the transfer function

Find the $z$-transform of both side of the state equation we get

$$zS(z) = AS(z) + BX(z)$$

From the output equation we have

$$Y(z) = CS(z) + DX(z)$$

Combine these two equations,

$$H(z) = \frac{Y(z)}{X(z)} = C(zI - A)^{-1}B + D$$
Example

• For the digital filters shown in the next page,
  – They have different state-space representations,
  – but they have the same transfer function:

\[
H(z) = \frac{\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2}}
\]

• This means that the two filters are actually the same one, with different realizations.
(a) \[ \begin{bmatrix} x_1(nT + T) \\ x_2(nT + T) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta_2 & -\beta_1 \end{bmatrix} \begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(nT) \]

\[ y(nT) = [\alpha_2 - \alpha_0 \beta_2 \quad \alpha_1 - \alpha_0 \beta_1] \begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix} + [\alpha_0] u(nT) \]

(b) \[ \begin{bmatrix} x_1(nT + T) \\ x_2(nT + T) \end{bmatrix} = \begin{bmatrix} -\beta_1 & 1 \\ -\beta_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix} + \begin{bmatrix} \alpha_1 - \alpha_0 \beta_1 \\ \alpha_2 - \alpha_0 \beta_2 \end{bmatrix} u(nT) \]

\[ y(nT) = [1 \quad 0] \begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix} + [\alpha_0] u(nT) \]
The equivalent transformation (等価変換)

- If $T$ is a non-singular matrix, we can get a new realization of the filter $DF(A,B,C,D)$ by an equivalent transform

  $$s'(n) = T^{-1} s(n)$$

- If we use the new state vector $s'$, the filter will become $DF(T^{-1}AT, T^{-1}B, CT, D)$.
- The transfer function of the new filter will be the same as that of the old one.
- Therefore, there can be infinitely many realizations for the same filter.
- We can find the “optimal” realization of the same system by selecting a suitable $T$, starting from a canonic form.
Why should we consider different realizations for the same system?

• Although the input-output relation is the same, one realization can be better than another when
  – There are computational errors.
  – The systems parameters changes.
• We say a system is **robust** if it is not sensitive to noises from both inside and outside.

制御工学には、モデルに不確かさが存在する状況においても一定の性能を維持する制御器の設計手法として、ロバスト制御という方法がある。

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wikipedia
Other realizations

- In practice, it is better to make many small modules, and realize large systems through cascade or parallel connection.
- The module often used is the 2\textsuperscript{nd} order filter realized in the bi-quad (or 2\textsuperscript{nd} order section) form.
The cascade realization (従続接続型)

The transfer function of any filter can be written in the following form

\[ H(z) = \prod_{i=1}^{M} C_i(z) \]

where \( C_i(z) \)'s are 1st or 2nd order sections.
The parallel realization (並列接続型)

For IIR filters, the transfer function can also be written in the following form:

\[ H(z) = \sum_{i=1}^{M} P_i(z) \]

where \( P_i(z) \) is a 1st or 2nd order section.
Merits of using modular systems

• Cascade connection
  – Good for pipeline computation → high through-put.
  – Less expensive components can be used in the modules close to the output.

• Parallel connection
  – Fast parallel computation.
  – No noise amplification → more robust.
Minimum norm form

• Among many realizations of a 2\textsuperscript{nd} order (biquad) filter, the minimum norm form has the minimum round-off noises and does not have overflow oscillation.

• For a 2\textsuperscript{nd} order filter, it has two poles given by

\[ p_i = r(\cos \theta \pm j \sin \theta) \]

\[ i = 1, 2 \text{ and } |r| < 1 \]
Minimum norm form

• The companion matrix of a minimum norm form is given below and the structure is shown in the figure.

\[
DF(A, B, C, d)
\]

\[
A = \begin{bmatrix}
  r \cos \theta & r \sin \theta \\
  -r \sin \theta & r \cos \theta 
\end{bmatrix}
\]
Homework

- A 3\textsuperscript{rd} order digital filter \( DF(A,B,C,D) \) is defined as follows:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-0.25 & 0 & 0.5 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
-0.25 & 0 & 0.5 \\
\end{bmatrix}, \quad D = [1]
\]

- Find the transfer function (伝達関数) of this filter.
- Discuss the stability (安定性) of this filter.

\textit{HINT: Find the difference equation first.}
Quiz and self-evaluation

• For the digital filter given in the previous page, draw the block diagram of the direct form II realization.

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