z-transformation - II
Topics of last lecture

• The z-transformation.
• Convergence region of z-transform.
• Relation between z-transform and Fourier transform.
• Relation between z-transform and difference equation.
• Transfer function (system function).
Topics of This lecture

• Rational function, zeros and poles.
• Stability of system again
• Inverse z-transformation
  – Partial-fraction expansion method
  – Contour integration (residue) method

Chapter 6 and Chapter 7
External representation

- $H(z)$, $H(e^{j\omega})$ and $h(n)$ are different external representations of an LTI system.
- Only the relation between the input and the output is given.
- Detailed information related to implementation is omitted.

I am ZHAO. If you do not believe, you may ask me any question.
Rational Function
有理関数 p. 92

• The transfer function of a system represented by difference equation is a rational function.

\[ y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k) \]

\[ Y(z) = -\sum_{k=1}^{N} a_k Y(z)z^{-k} + \sum_{k=0}^{N} b_k X(z)z^{-k} \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \]

Numerator is a polynomial
Denominator is also a polynomial
Zeros and poles

• The roots of the denominator are called poles (極).
• The roots of the numerator are called zeros (零点).
• From the *fundamental theorem of algebra*, we have

\[
H(z) = \frac{H_0 \prod_{k=1}^{N} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}
\]

where the zeros and poles are complex numbers.
Zeros and poles

- Example 6.3 in pp. 92-93.

```matlab
Nz=[1 2 1];
Dz=[1 -1 0.75];
zr=roots(Nz);
pl=roots(Dz);
zplane(Nz,Dz);
legend('zeros','poles');
```
Stability of LTI system

• Theorem 7.1 (p. 113): An LTI system is stable if and only if its poles are inside the unit circle. In other words, $|p_k|<1$ for $k=1,2,\ldots,N$.

• This can be proved based on the definitions of the z-transform and the transfer function.

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} = \frac{H_0 \prod_{k=1}^{N} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$
Region of convergence of $H(z)$

- Since $|H(z)|$ is infinitely large at any pole, the region of convergence cannot contain any pole.
H(z) exists for z larger than the poles

- If H(z) exists at \( z_0 \), H(z) will exist for all z satisfying \(|z|>|z_0|\), because

\[
\sum_{n=0}^{\infty} |h(n)z^{-n}| < \infty \\
\sum_{n=0}^{\infty} \left| h(n)z_0^{-n} \right| < \infty
\]
The poles must be inside the unit circle

• An LTI system is stable iff $H(z)$ exists for $|z|=1$.
• That is, the frequency response $H(e^{j\omega})$ exists.
• This is equivalent to

$$
\sum_{n=0}^{\infty} |h(n)e^{j\omega}| = \sum_{n=0}^{\infty} |h(n)| < \infty
$$

• Thus, an LTI system is stable iff all poles are in the unit circle.
Example

• Suppose the transfer function of an LTI system is given below. Answer the following questions:
  – What are the “zeros” of this system?
  – What are the “poles” of the system?
  – Is this system stable (yes/not)?

\[ H(z) = \frac{0.5(z - 1)z}{(z - 0.1)(z - 1.5)} \]
Example 7.6 (p. 114)

```matlab
b=[1 0 1];
a=[1 -0.9 0.81];
zr=roots(b)
pl=roots(a)
zplane(b,a);
legend('zeros','poles');
Maxpl=max(abs(pl))
if Maxpl < 1
    disp('Stable');
else
    disp('Unstable');
end
```
The inverse z-transformation
pp. 97-100

• Definition of the inverse z-transformation

\[ x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1}dz \] (6.45)

– where \( \Gamma \) is a counterclockwise closed contour in the region of convergence encircling the origin of the z-plane.

• Inverse z-transform is important to find \( h(n) \) from \( H(z) \).
Methods for finding the inverse z-transformation

• Partial-fraction expansion method (部分分数展開法)
• Power series expansion method （べき級数展開法）
• Contour integration (residue) method （周回積分法）

Here we consider only $X(z)$ represented as a rational function
Partial fraction expansion method

• The basic idea is to divide $X(z)$ into many simple functions as follows:

$$X(z) = \sum_{i=1}^{k} X_i(z)$$

• The inverse z-transformation is then found

$$x(n) = \sum_{i=1}^{k} Z^{-1}[X_i(z)]$$
**Example**

\[
X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})} = \frac{Az}{z-1} + \frac{Bz}{z-e^{-aT}}
\]

\[
X(z)(z-1) = \frac{(1-e^{-aT})z}{z-e^{-aT}} = Az + \frac{Bz(z-1)}{z-e^{-aT}}
\]

\[
X(z)(z-e^{-aT}) = \frac{(1-e^{-aT})z}{z-1} = \frac{Az(z-e^{-aT})}{z-1} + Bz
\]

\[
z = 1 \Rightarrow A \cdot 1 = \frac{(1-e^{-aT}) \cdot 1}{1-e^{-aT}} \Rightarrow A = 1
\]

\[
z = e^{-aT} \Rightarrow Be^{-aT} = \frac{(1-e^{-aT})e^{-aT}}{e^{-aT}-1} \Rightarrow B = -1
\]

\[
X(z) = \frac{z}{z-1} - \frac{z}{z-e^{-aT}}
\]
Contour integration method

- This method is based on the following Cauchy’s residue theorem

\[
x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1} \, dz
\]

\[
= \sum \text{[residue of } X(z)z^{n-1} \text{]} \quad \text{for all poles}
\]
Calculation of the residues

• If \( a \) is a first order pole, the corresponding residue is given by

\[
\text{Res}(a) = \lim_{z \to a} (z - a) X(z)z^{n-1}
\]

• If \( a \) is a \( m \)-th order pole, the residue is

\[
\text{Res}(a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}[(z - a)^m X(z)z^{n-1}]
\]
Example

• The poles of $X(z)$ are 1 and $e^{-at}$, and both of them are first order poles.

• The residues of the poles are given by

\[
\text{Res}(1) = \lim_{z \to 1} \left[ (z - 1) \frac{(1 - e^{-aT})z}{(z - 1)(z - e^{-aT})} z^{n-1} \right] = \frac{(1 - e^{-aT}) \cdot 1}{(1 - e^{-aT})} 1^{n-1} = 1
\]

\[
\text{Res}(e^{-aT}) = \lim_{z \to e^{-aT}} \left[ (z - e^{-aT}) \frac{(1 - e^{-aT})z}{(z - 1)(z - e^{-aT})} z^{n-1} \right] = -e^{-anT}
\]

\[
\therefore x(n) = \text{Res}(1) + \text{Res}(e^{-aT}) = 1 - e^{-anT}
\]
Homework

• Read Example 6.10 in p. 99.
• Find the original signal from the following z-transform using Contour integration method:

\[ X(z) = \frac{(2z - 1)z}{2(z - 1)(z + 0.5)} \]

• Verify your answer using the following Matlab program (see the program given in p. 100):

```matlab
syms z;
X=(2*z-1)*z/(2*(z-1)*(z+0.5));
x=iztrans(X)
```
Quiz and self-evaluation

The transfer function of an LTI system is given as follows:

\[ H(z) = \frac{z}{(z - 0.1)(z - 0.2)} \]

1. Find the zeros and poles.

2. Discuss about the stability of this system based on the poles.

Name:_________________________ Student ID:_________________________. 

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The transfer function is given by:

\[ H(z) = \frac{z}{(z - 0.1)(z - 0.2)} \]

1. **Zeros and Poles:**
   - Zeros: \( z = 0 \) (due to numerator)
   - Poles: \( z = 0.1, 0.2 \) (due to denominator)

2. **Stability:**
   - The system is **stable** since all poles are inside the unit circle (\( |z| < 1 \)).