Linear and Time Invariant Systems – I
Topics of last lecture

• Signals: Speech, audio, light, radio, TV, radar, etc.
• Signal processing: Enhancement, restoration, reconstruction, synthesis, estimation, etc.
• Signal representation: Continues, discrete, and digital
• Fundamental signals: Unit sample, unit step, exponential, sinusoid
• System representation: A functional module between an input signal and an output signal
Topics of this lecture

• Linear and time-invariant system
• Impulse response
• Convolution sum
• Connection methods
• Causal system
• Stable system

Chapter 5, start from page 66 of textbook
The output \( y(n) \) is found using the same linear combination as that used for finding the input \( x(n) \). This is called superposition principle (重ね合わせの原理).
Time-invariant system (時不変システム)

\[ y(n) = S \left[ x(n) \right] \]
\[ y(n - n_0) = S \left[ x(n - n_0) \right] \]
Example: Digital filter is a kind of system for signal processing. Suppose that the LTI digital filter is defined by (a) or (b), try to discuss its linearity ($T=1$).

(a) $y(nT) = \phi[x(nT)] = 5x^2(nT-T)$

(b) $y(nT) = \phi[x(nT)] = (nT)^2x(nT-2T)$

解 (a) $\phi[ax(nT)] = 5a^2x^2(nT-T)$

ところで

$$a\phi[x(nT)] = 5ax^2(nT-T)$$

が得られる。このとき

$$\phi[ax(nT)] = a\phi[x(nT)]$$

となる。したがって、このフィルタは非線形である。

(b) $\phi[ax_1(nT) + bx_2(nT)] = (nT)^2[ax_1(nT-2T) + bx_2(nT-2T)]$

$$= a(nT)^2x_1(nT-2T) + b(nT)^2x_2(nT-2T)$$

$$= a\phi[x_1(nT)] + b\phi[x_2(nT)]$$

→ このフィルタは線形である。
Example: For the system given by (a) or (b), try to discuss if the system is time-invariant or not.


taken from the reference book

(a) \( y(nT) = \phi[x(nT)] = 4nTx(nT) \)

(b) \( y(nT) = \phi[x(nT)] = 5x(nT-T) + 10x(nT-2T) \)

解 (a) \( \phi[x(nT-n_0T)] = 4nTx(nT-n_0T) \)

ところで

\[ y(nT-n_0T) = 4(nT-n_0T)x(nT-n_0T) \]

このとき

\[ \phi[x(nT-n_0T)] \neq y(nT-n_0T) \]

となる。したがって、このフィルタは時変である。

(b) \( \phi[x(nT-n_0T)] = 5x(nT-n_0T-T) + 10x(nT-n_0T-2T) \)

\[ = y(nT-n_0T) \]

明らかにこのフィルタは時不変である。
Impulse response (インパルス応答)

\[
\delta(n) = \begin{cases} 
1 & n=0 \\
0 & n\neq0 
\end{cases}
\]

This function is called the \textit{impulse signal}, unit sample signal, delta signal, and others (see pp. 13-14 in the text).

The output (response) of a system to the impulse signal is called the \textit{impulse response} （インパルス応答）.
Response of an LTI system (線形時不変システムの応答)

\[ x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n - k) \]

where

\[ \delta(n - k) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} \]

suppose that the impulse response is \( h(n) \). Since the system is time-invariant,

\[ S[\delta(n)] = h(n) \rightarrow S[\delta(n - k)] = h(n - k) \]

with these equations, we can find the output as follows.
Convolution sum (畳み込み和)

\[ y(n) = S[x(n)] \]
\[ = S\left[ \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \right] \]
\[ = \sum_{k=-\infty}^{\infty} x(k)S[\delta(n-k)] \]
\[ = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \]

\[ y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) \ast h(n) \]

**Linearity of the system**

**Time-invariance of the system**

\[ x(n) \rightarrow h(n) \rightarrow y(n) = x(n) \ast h(n) \]
Importance of impulse response

• The impulse response $h(n)$ can determine the characteristics of an LTI system completely.
• For any input $x(n)$, the output $y(n)$ is the convolution sum of $x(n)$ and $h(n)$.
• Convolution sum is commutative (可換), that is

\[
y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n) \\
= \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n)
\]
Classification of systems based on the impulse response

- Digital filters can be divided into two categories based on the length (duration) of the impulse response.
- If the impulse response is finite length, the system is called an FIR (Finite Impulse Response) system or FIR filter.
- If the length of the impulse response is infinite, it is an IIR (Infinite Impulse Response) system, or IIR filter.
Example 5.2 pp. 70-71

• Suppose that the impulse response of a digital filter is given below:

\[ h(n) = \alpha^n \cdot u_0(n) \quad u_0(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

✓ Represent the input-output relation using convolution sum;
✓ Find the response of the filter for the unit step signal \( u_0(n) \).
Example 5.2
Cascade connection of LTI systems
(線形時不変システムの縦続接続)

\[
x(n) \rightarrow h_1(n) \rightarrow h_2(n) \rightarrow y(n)
\]

\[
x(n) \rightarrow h_2(n) \rightarrow h_1(n) \rightarrow y(n)
\]

\[
x(n) \rightarrow h_1(n) * h_2(n) \rightarrow y(n)
\]
Parallel connection of LTI systems
(線形時不変システムの並列接続)

\[ x(n) \xrightarrow{h_1(n)} y(n) \]

\[ x(n) \xrightarrow{h_1(n) + h_2(n)} y(n) \]
Stable systems (安定的システム)

• A system is stable iff

\[
\sum_{k=-\infty}^{\infty} | h(k) | < \infty
\]

• This is the so called bounded-input-bounded output (BIBO) stability condition (有界入力有界出力).
• The system can give meaningful output only if it is stable.
Example 5.3

- Try to discuss the stability of the system given in Example 5.2.
Causal systems (因果的システム)

- A causal system is a physically realizable (物理的に実現可能な) system (not for off-line processing).
- An LTI system is causal if and only if (iff: 必要十分条件)
  \[ h(n) = 0 \quad \text{for} \quad n < 0 \]
- A signal is called causal iff \( x(n) = 0 \) for \( n < 0 \).
Causal systems

• A system is causal iff the current output \( y(n) \) is calculated based only on already observed data \( x(n), x(n-1), x(n-2), \ldots \)

\[
y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)
\]

\[
= \sum_{k=-\infty}^{\infty} x(n-k)h(k) = \sum_{k=0}^{\infty} x(n-k)h(k)
\]
Homework-1

• Suppose that the impulse response of a digital filter is given by

\[ h(n) = a^n u_0(n) \]

• Discuss the stability and causality of the filter.
Suppose that the impulse response of a digital filter is given by

\[ h(n) = e^{n\alpha} u_0(n) \]

find the output \( y(n) \) of the filter using convolution sum when the input \( x(n) \) is defined by \( x(0)=1, x(1)=1, x(2)=1, x(3)=1, \) and \( x(n)=0, \) for \( n>3. \)
Quiz and self-evaluation

• For the system given in homework-2, tell if it is causal or not.

• For the same system, give the condition for the system to be stable.