Automata and Languages

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Nondeterministic Finite Automata with empty moves (λ-NFA)

Definition

A nondeterministic finite automaton with empty moves (λ-NFA) M is defined by a 5-tuple $M=(Q,\Sigma,\delta,q_0,F)$, with

- Q: finite set of states
- Σ: finite input alphabet
- δ: transition function $\delta: Q \times (\Sigma \cup \{\lambda\}) \rightarrow P(Q)$
- $q_0 \in Q$: start state
- $F \subseteq Q$: set of final states

Nondeterministic Finite Automata with empty moves (λ-NFA)

Definition

A string w is ***accepted*** by a λ-NFA M if and only if *there exists* a path starting at q_0 which is labeled by w and ends in a final state.

The ***language accepted by*** a λ-NFA M is the set of all strings which are accepted by M and is denoted by $L(M)$.

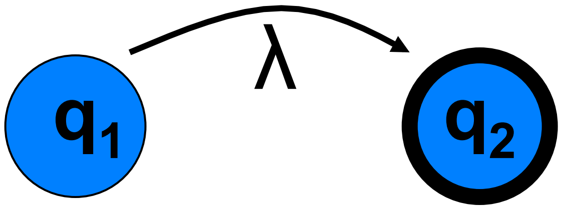
$$L(M)=\{w: \delta(q_0,w) \cap F \neq \Phi\}$$

Nondeterministic Finite Automata with empty moves (λ-NFA)

Notes

$$\delta : Q \times (\Sigma \cup \{\lambda\}) \rightarrow P(Q)$$

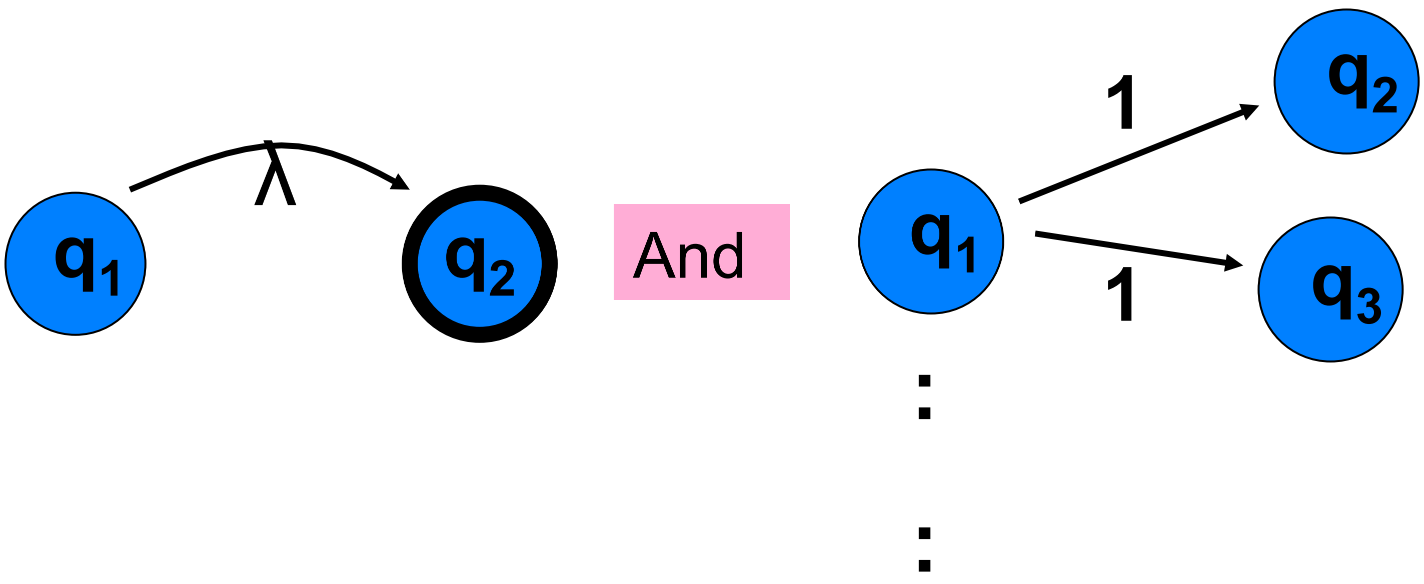
- A *λ-transition* causes the machine to change its state non-deterministically, without consuming any input.



Nondeterministic Finite Automata with empty moves (λ-NFA)

Notes

A λ-NFA has transition rules/possibilities like:



Empty string transition

Nondeterministic transition

Nondeterministic Finite Automata with empty moves (λ-NFA)

Nondeterminism ~ Parallelism

For any string w, the nondeterministic automaton can be in a subset $\subseteq Q$ of several possible states.

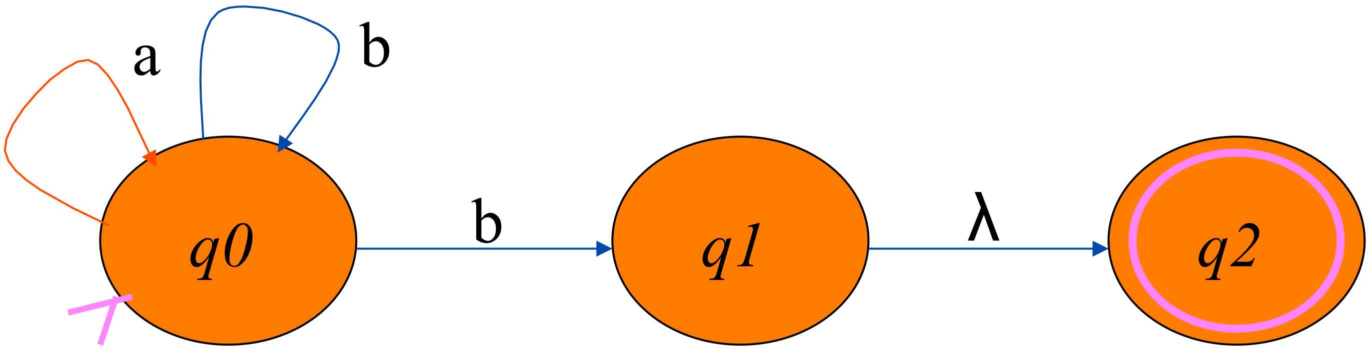
If the final set contains a final state, then the automaton accepts the string.

"The automaton processes the input in a parallel fashion; its computational path is no longer a line, but more like a tree".

Nondeterministic Finite Automata with empty moves (λ-NFA)

We can write the NFA in two ways

1. State digraph



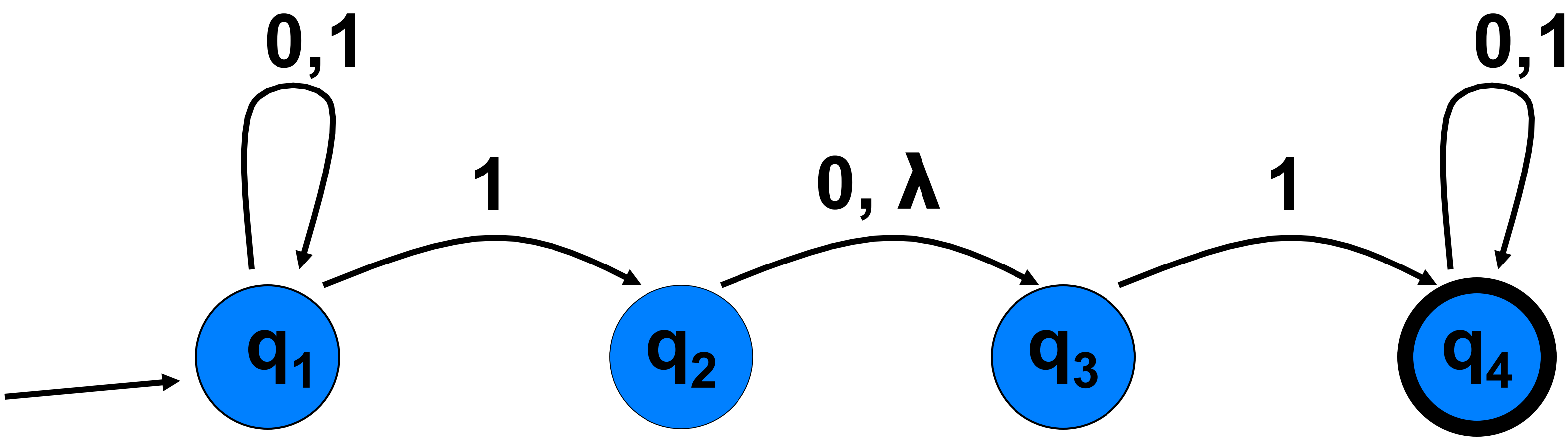
2. Table

$$\delta : Q \times (\Sigma \cup \{\lambda\}) \rightarrow P(Q)$$

δ	a	b	λ
q0	{q0}	{q0,q1}	ϕ
q1	ϕ	ϕ	{q2}
q2	ϕ	ϕ	ϕ

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example

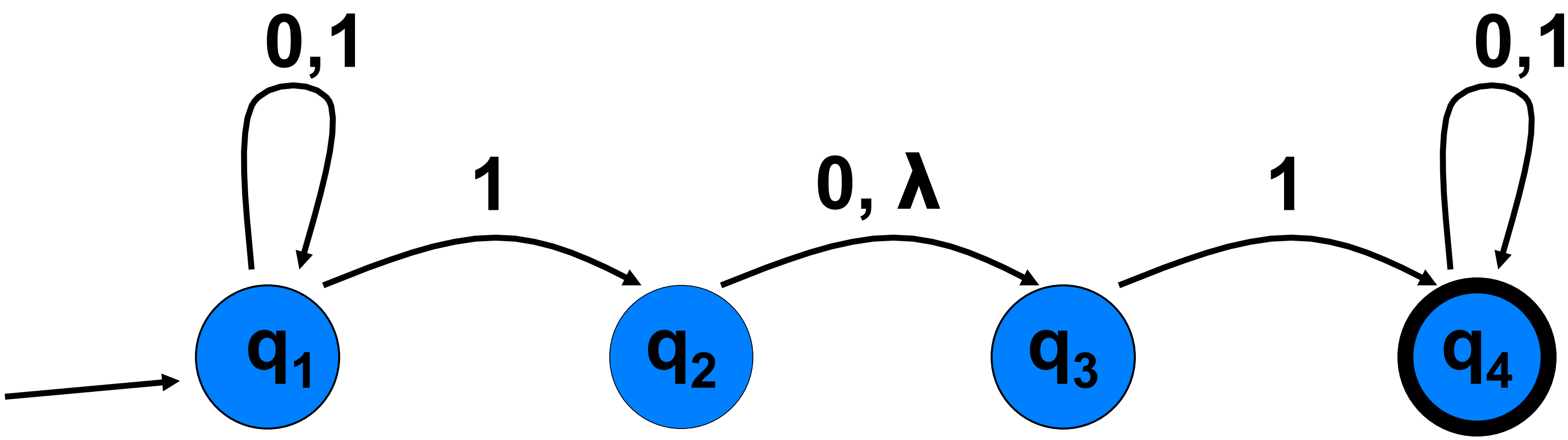


This automaton accepts "0110", because there is a *possible* path that leads to a final state,
namely: $q_1 \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_4 \rightarrow q_4$

(note that $q_1 \rightarrow q_1 \rightarrow q_1 \rightarrow q_1 \rightarrow q_1$ is *not* accepting)

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example

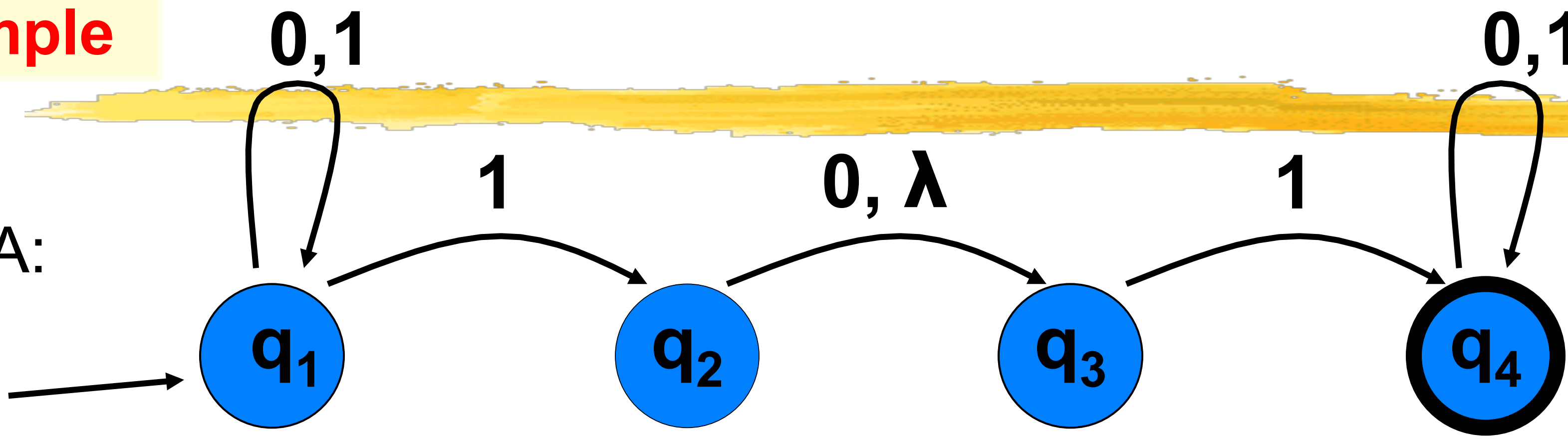


The string 1 gets rejected: on “1” the automaton can only reach: $\{q_1,q_2,q_3\}$.

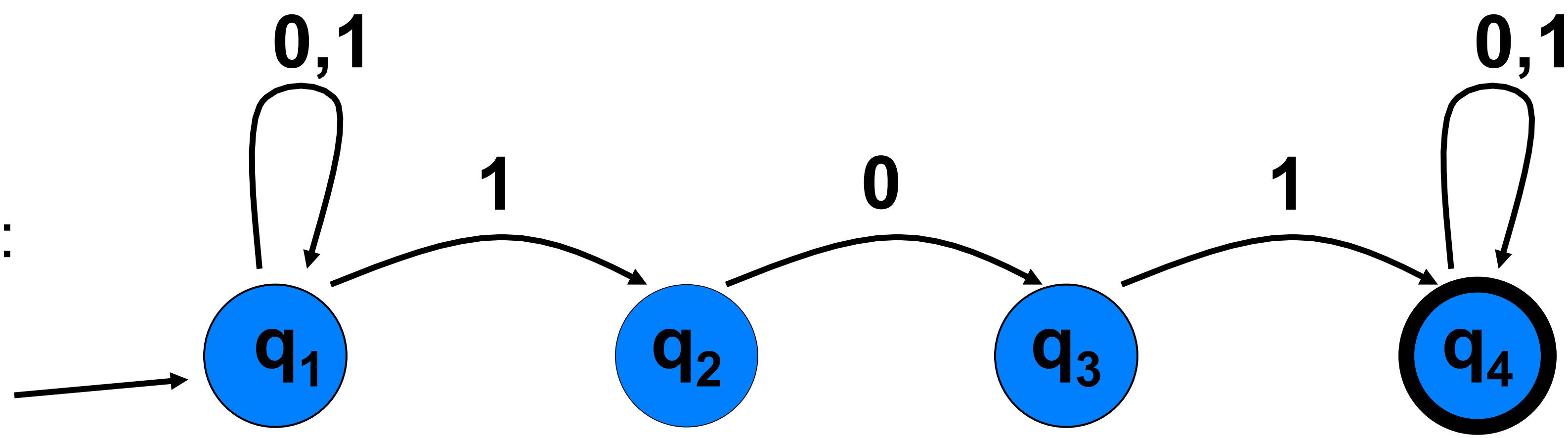
Difference between NFA and λ-NFA

Example

λ-NFA:



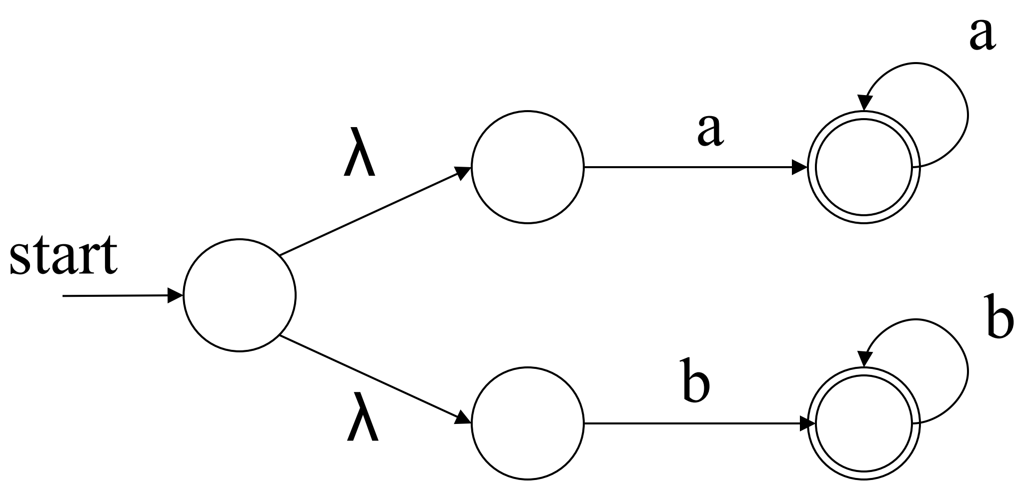
NFA:



The string 11 is accepted by the above λ-NFA
And rejected by the above NFA

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example



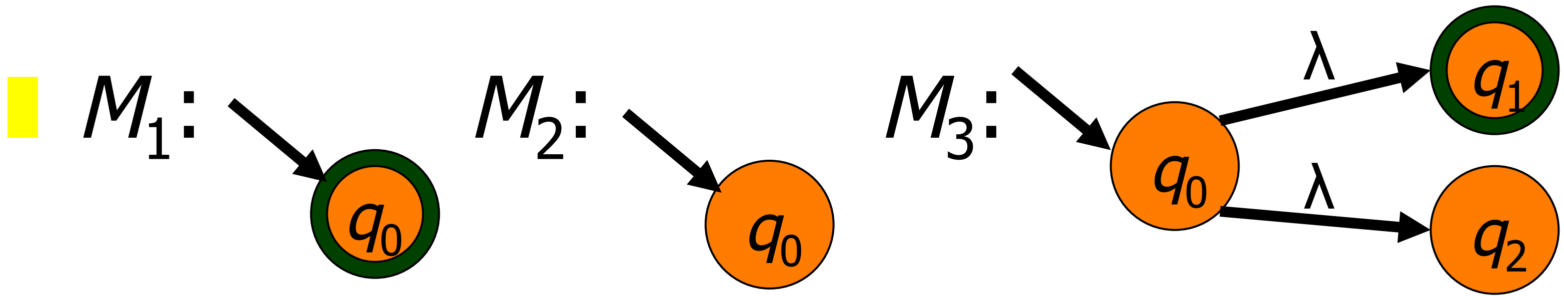
A λ-transition is taken without consuming any character from the input.

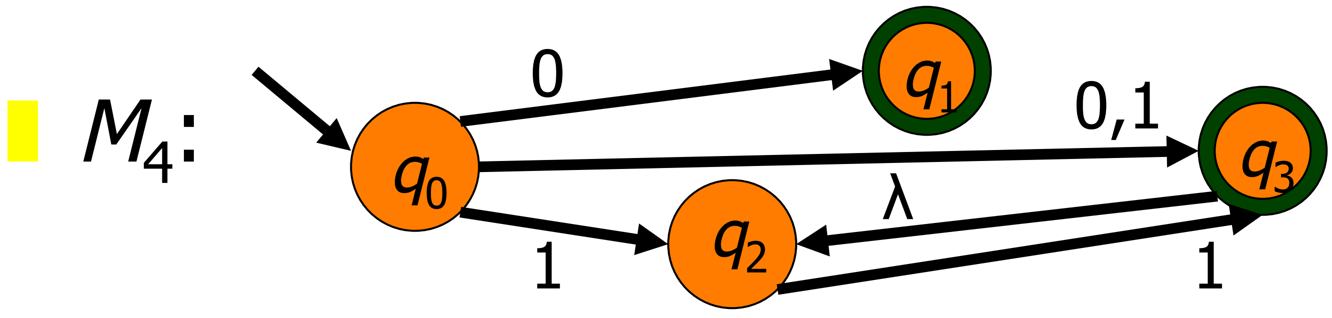
What does the NFA above accepts?

aa*+bb*

Nondeterministic Finite Automata with empty moves (λ-NFA)

Quiz

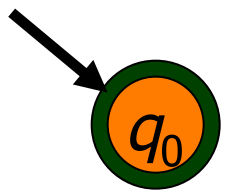


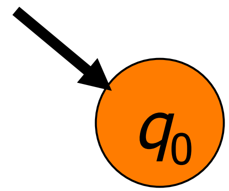


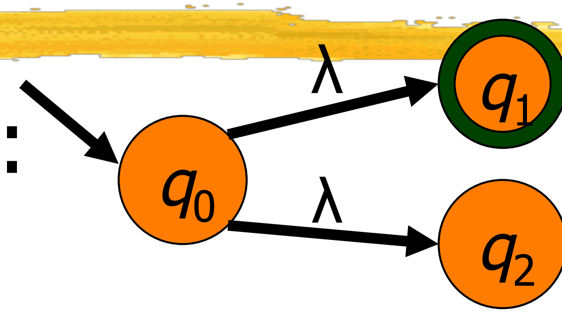
What are $\delta(q_0,0)$, $\delta(q_0,1)$, $\delta(q_0, \lambda)$ in each of M_1, M_2, M_3 and in M_4.?

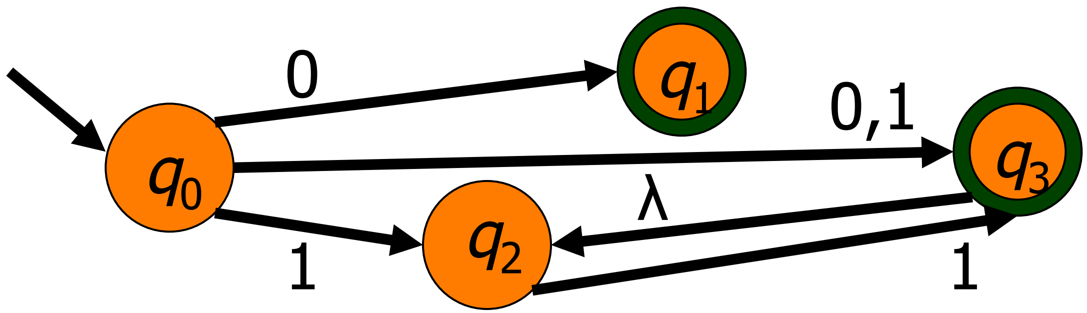
Nondeterministic Finite Automata with empty moves (λ-NFA)

Answer

- *M*1: 

  *M*2: 

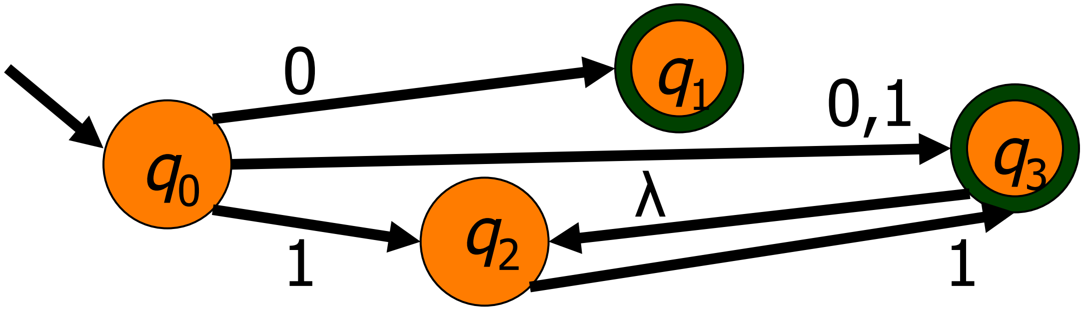
  *M*3: 

- *M*4: 

$M1$: $\delta(q_0,0) = \delta(q_0,1) = \delta(q_0, \lambda) = \varnothing$
$M2$: Same
$M3$: $\delta(q_0,0) = \delta(q_0,1) = \varnothing$, $\delta(q_0, \lambda) = \{q_1,q_2\}$
$M4$: $\delta(q_0,0)=\{q_1,q_3\}$, $\delta(q_0,1)=\{q_2,q_3\}$, $\delta(q_0, \lambda)=\varnothing$

Nondeterministic Finite Automata with empty moves (λ-NFA)

Quiz

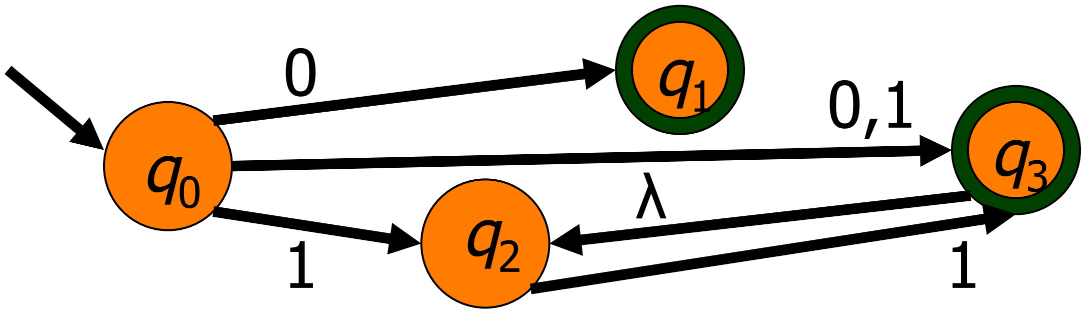


Which of the following strings is accepted?

1. λ
2. 0
3. 1
4. 0111

Nondeterministic Finite Automata with empty moves (λ-NFA)

Answer



1. λ is rejected. No path labeled by empty string from start state to an accept state.
2. 0 is accepted. EG the path $q_0 \xrightarrow[0]{} q_1$
3. 1 is accepted. EG the path $q_0 \xrightarrow[1]{} q_3$
4. 0111 is accepted. There is only one accepted path:

$$q_0 \xrightarrow[0]{} q_3 \xrightarrow[\lambda]{} q_2 \xrightarrow[1]{} q_3 \xrightarrow[\lambda]{} q_2 \xrightarrow[1]{} q_3 \xrightarrow[\lambda]{} q_2 \xrightarrow[1]{} q_3$$

Nondeterministic Finite Automata with empty moves (λ-NFA)

Definition

Given a λ-NFA state **s**, the **λ-closure(s)** is the set of states that are reachable through λ-transition from **s**.

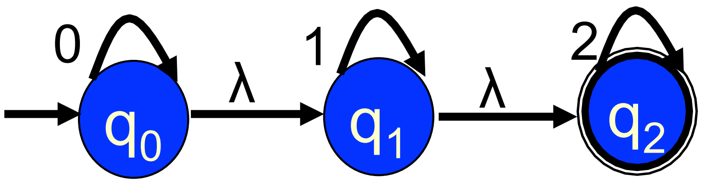
λ-closure(s)={q: there is a path from s to q labeled λ}

Given a set of λ-NFA states **T**, the **λ-closure(T)** is the set of states that are reachable through λ-transition from any state **s** $\in$**T**.

$$\lambda\text{-closure}(T)=\bigcup_{s\in T} \lambda\text{-closure}(s)$$

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example 1:



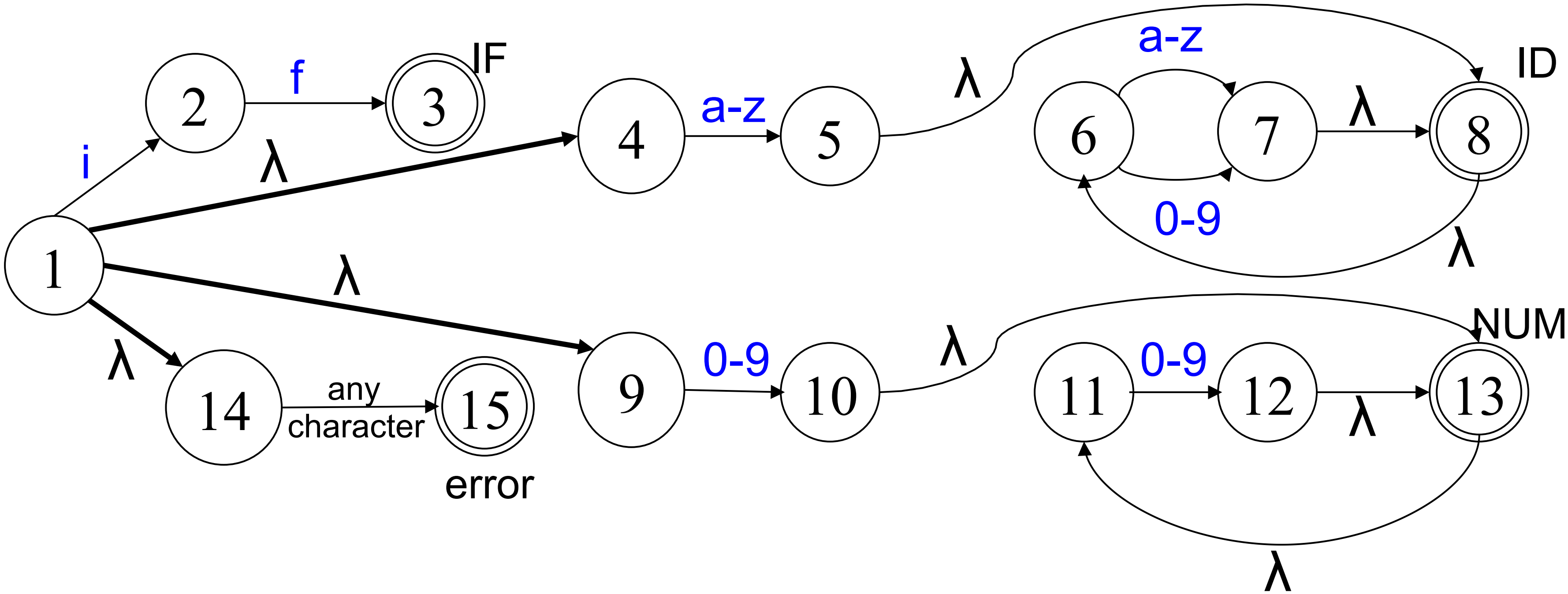
λ-closure(q_0) = {q_0, q_1, q_2}

λ-closure(q_1) = {q_1, q_2}

λ-closure(q_2) = {q_2}

Nondeterministic Finite Automata with empty moves (λ-NFA)

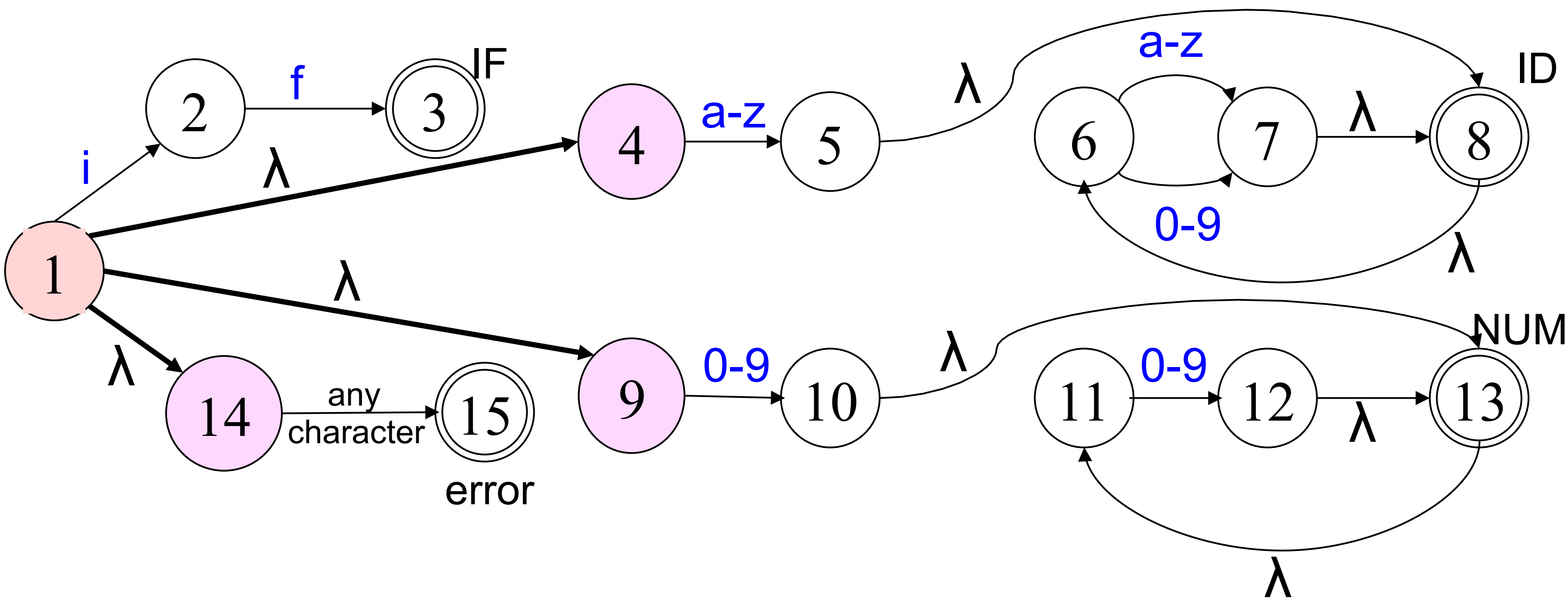
Example 2:



What states can be reached from state 1 without consuming a character?

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example

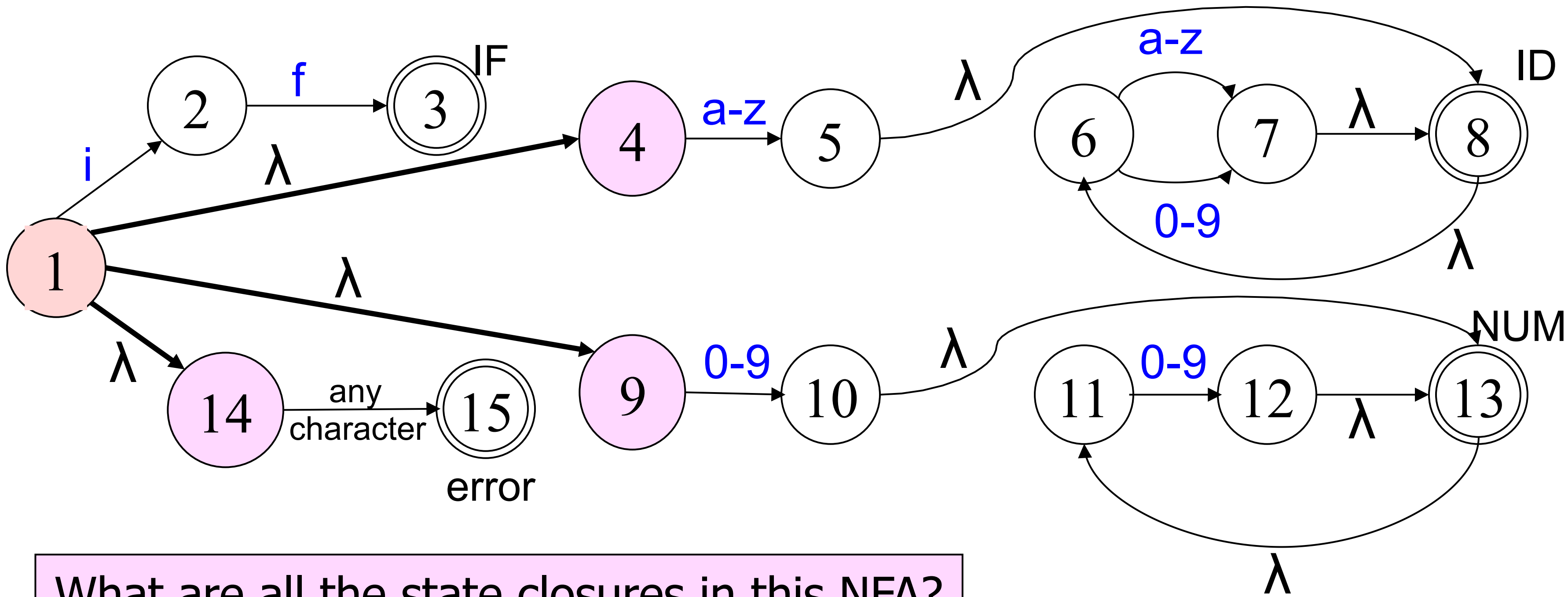


What states can be reached from state 1 without consuming a character?

{1,4,9,14} form the **λ-closure** of state 1

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example



What are all the state closures in this NFA?

closure(1) = {1,4,9,14}
closure(5) = {5,6,8}
closure(8) = {6,8}
closure(7) = {6,7,8}

closure(10) = {10,11,13}
closure(13) = {11,13}
closure(12) = {11,12,13}

Nondeterministic Finite Automata with empty moves (λ-NFA)

Definition: Extension of δ

$$\delta : Q \times (\sum \cup \{\lambda\}) \rightarrow P(Q) \quad \Longrightarrow \quad \hat{\delta} : Q \times \sum^{*} \rightarrow P(Q)$$

$\hat{\delta}$ **is defined as follows:**

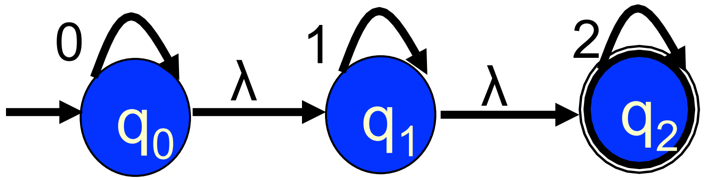
1. $\hat{\delta}$**(q,λ)=λ-closure(q)**

2. $\hat{\delta}$ (q,wa)=**λ-closure(T)** **where**

T={p: p∈δ(r,a) and r∈ $\hat{\delta}$(q,w)}, a∈**Σ, w**∈Σ*

Nondeterministic Finite Automata with empty moves (λ-NFA)

Example: Extension of δ



$\hat{\delta}(q_0, 01) = \{q_1, q_2\}$

λ-NFA → NFA

- <u>Theorem</u>: For every language L that is accepted by a λ-NFA, there is an NFA that accepts L as well.

- λ-NFA and NFA are equivalent computational models.

λ-NFA → NFA

- Proof:

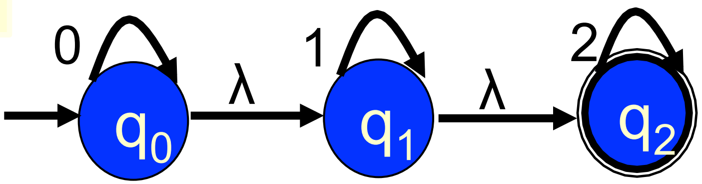
Let $M=(Q,\mathbf{\Sigma},\delta,q_0,F)$ be a λ-NFA, an equivalent NFA, $M'=(Q,\mathbf{\Sigma},\delta',q_0,F')$ can be constructed as follows:

1. $$F' = \begin{cases} F \cup \{q_0\} & \text{If } \lambda\text{-closure}(q_0) \cap F \neq \Phi \\ F & \text{Otherwise} \end{cases}$$

2. $$\delta'(q,a) = \hat{\delta}(q,a)$$

λ-NFA → NFA

Example:

For the λ-NFA:  Construct the equivalent NFA?

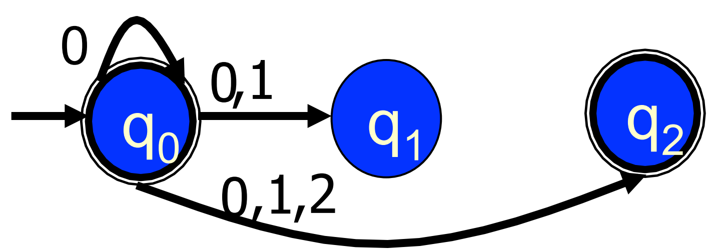
Answer:

Given λ-NFA		Constructed NFA
$Q=\{q_0,q_1,q_2\}$ and $\Sigma=\{0,1,2\}$	⟹	$Q=\{q_0,q_1,q_2\}$ and $\Sigma=\{0,1,2\}$
λ-closure$(q_0)=\{q_0,q_1,q_2\} \cap F \neq \Phi$	⟹	$F' = \{q_0,q_2\}$

$$\delta'(q_0,0) = \hat{\delta}(q_0,0) = \{q_0,q_1,q_2\}$$

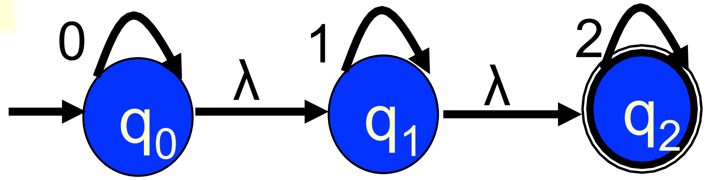
$$\delta'(q_0,1) = \hat{\delta}(q_0,1) = \{q_1,q_2\}$$

$$\delta'(q_0,2) = \hat{\delta}(q_0,2) = \{q_2\}$$



λ-NFA → NFA

Example:

For the λ-NFA: 

Construct the equivalent NFA?

Answer:

Given λ-NFA

$Q=\{q_0,q_1,q_2\}$ and $\Sigma=\{0,1,2\}$ ⟹

λ-closure$(q_0)=\{q_0,q_1,q_2\} \cap F \neq \Phi$ ⟹

$$\delta'(q_1,0)=\hat{\delta}(q_1,0)=\Phi$$

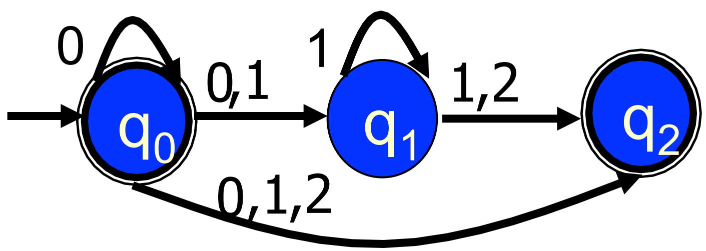
$$\delta'(q_1,1)=\hat{\delta}(q_1,1)=\{q_1,q_2\}$$

$$\delta'(q_1,2)=\hat{\delta}(q_1,2)=\{q_2\}$$

Constructed NFA

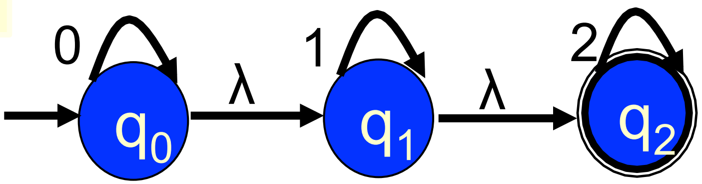
$Q=\{q_0,q_1,q_2\}$ and $\Sigma=\{0,1,2\}$

$F' = \{q_0,q_1\}$



λ-NFA → NFA

Example:

For the λ-NFA:  Construct the equivalent NFA?

Answer:

Given λ-NFA

$Q=\{q_0,q_1,q_2\}$ and $\mathbf{\Sigma}=\{0,1,2\}$ ⟹

$\lambda\text{-closure}(q_0)=\{q_0,q_1,q_2\} \cap F \neq \Phi$ ⟹

$$\delta'(q_2,0) = \hat{\delta}(q_2,0) = \Phi$$

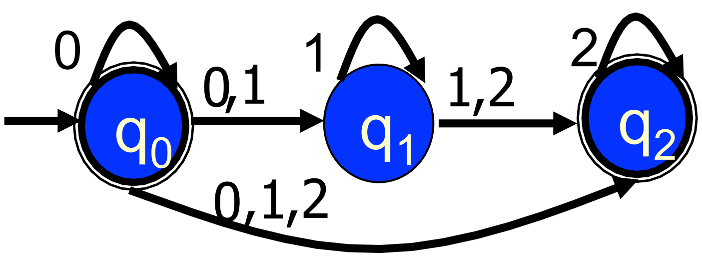
$$\delta'(q_2,1) = \hat{\delta}(q_2,1) = \Phi$$

$$\delta'(q_2,2) = \hat{\delta}(q_2,2) = \{q_2\}$$

Constructed NFA

$Q=\{q_0,q_1,q_2\}$ and $\Sigma=\{0,1,2\}$

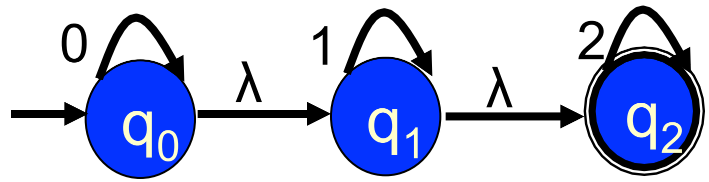
$F' = \{q_0,q_1\}$



λ-NFA → NFA

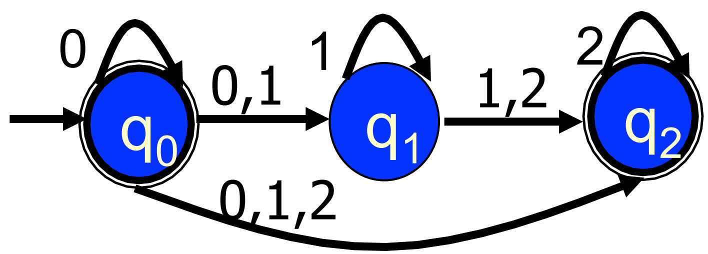
Example:

For the λ-NFA:



Construct the equivalent NFA?

Answer:



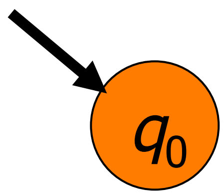
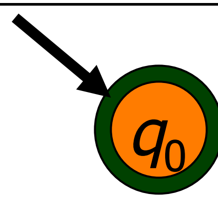
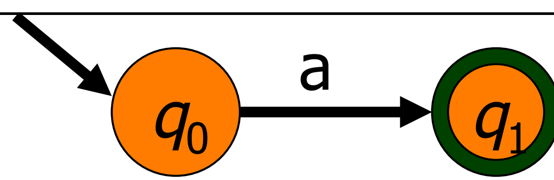
RE → λ-NFA

- Theorem: Let r be RE, there exist a λ-NFA that accepts L(r).

RE → λ-NFA

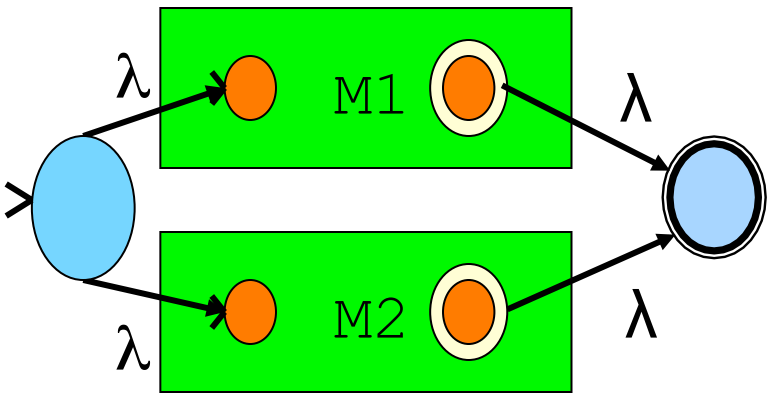
Proof:

The proof works by induction, using the recursive definition of regular expressions.

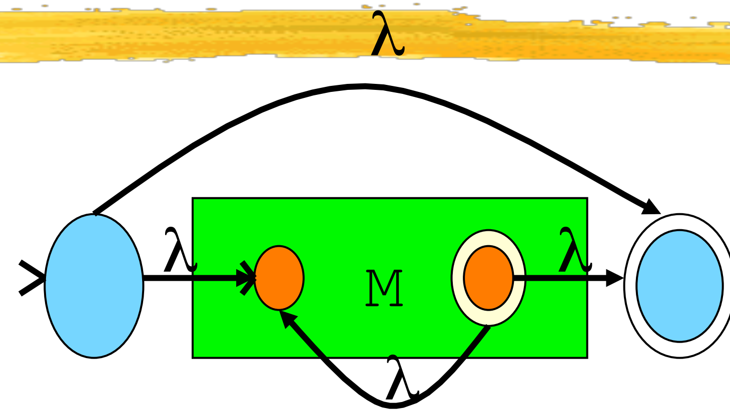
RE	λ-NFA
Φ	
λ	
a	

RE → λ-NFA

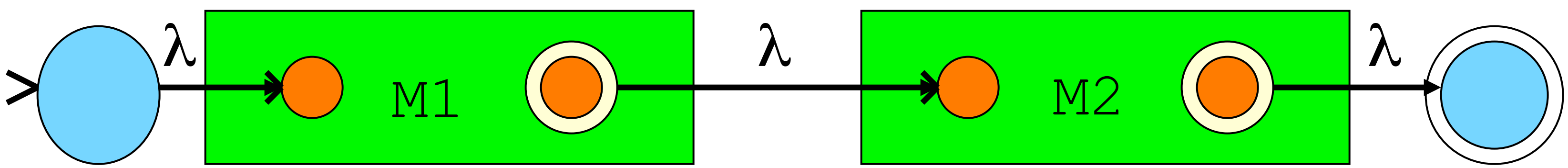
Proof:



r1+r2 → *L*(M1) ∪ *L*(M2)



*r** → *L*(M)*



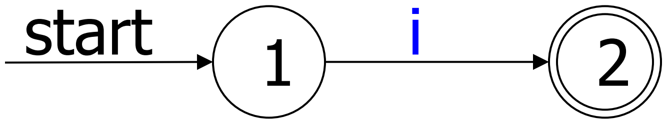
r1.r2 → *L*(M1) *L*(M2)

RE → λ-NFA

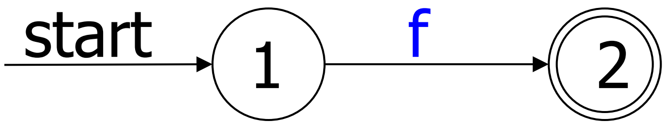
Example 1

For the regular expression r=if we build the λ-NFA as follows:

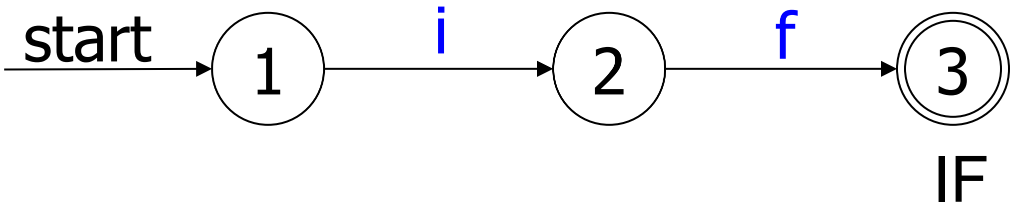
The λ-NFA for a symbol i is:



The λ-NFA for a symbol f is:



The λ-NFA for the regular expression if is:

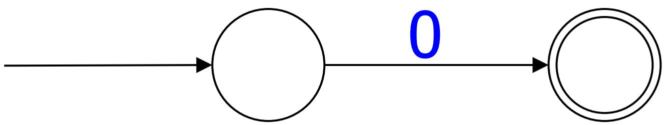


RE → λ-NFA

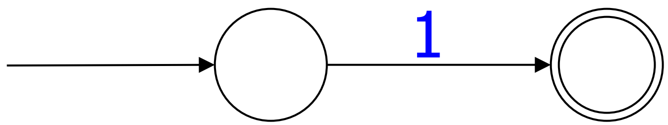
Example 2

For the regular expression r=0+1* build the equivalent λ-NFA ?

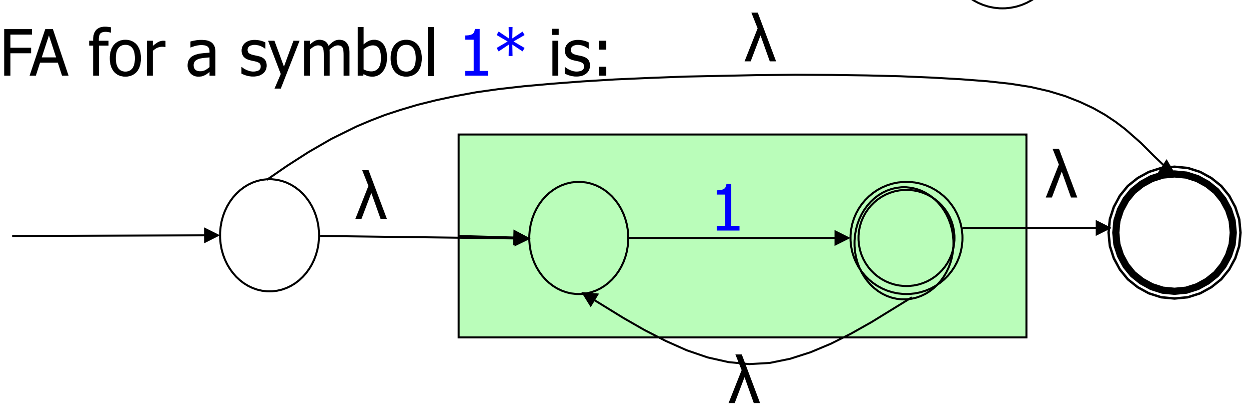
The λ-NFA for a symbol 0 is:



The λ-NFA for a symbol 1 is:



The λ-NFA for a symbol 1* is:



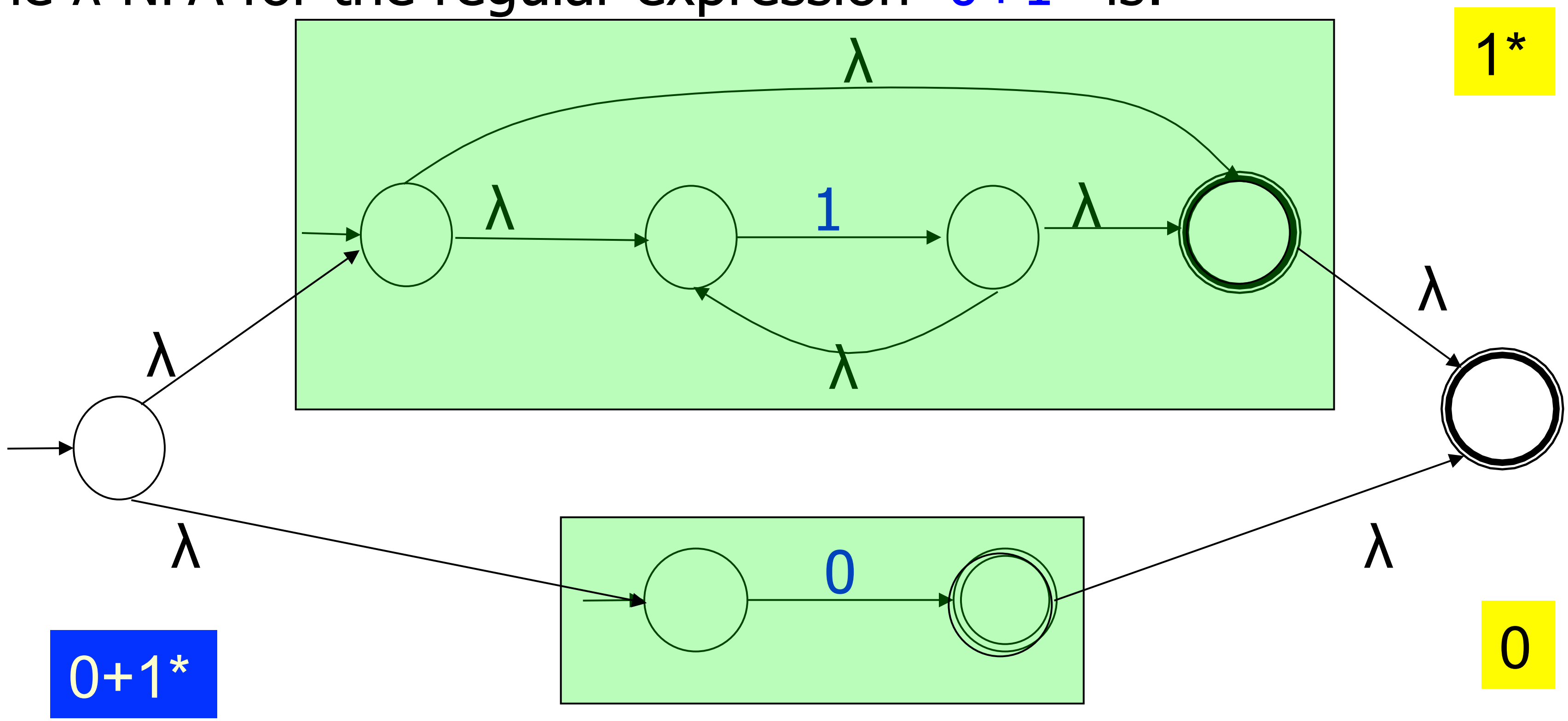
The λ-NFA for the regular expression 0+1* is:

RE → λ-NFA

Example 2

For the regular expression r=0+1* build the equivalent λ-NFA ?

The λ-NFA for the regular expression 0+1* is:



RE → λ-NFA

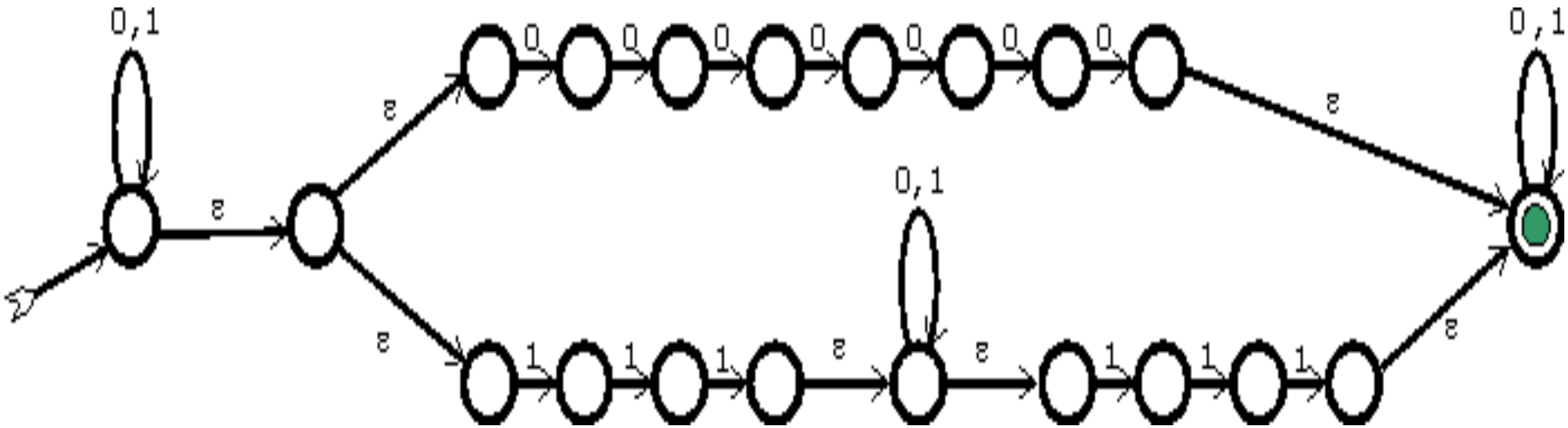
Example 3

Q: Find an NFA for the regular expression
$(0\cup 1)^*(0000000\cup 111(0\cup 1)^*111)(0\cup 1)^*$

RE → λ-NFA

Example 3

(0∪1)*(0000000∪111(0∪1)*111)(0∪1)*



Note that: in this example ε = λ

RE → λ-NFA

Exercise

Construct a λ-NFA for the regular expression:

010*1+(1+0)*+101*