Automata and Languages

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Mathematical Background

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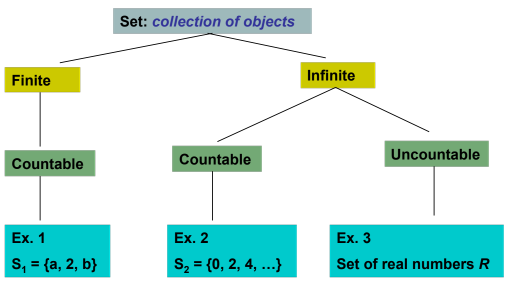
- Sets
- Relations
- Functions
- Graphs
- Proof techniques

Sets



∈-Notation & ⊆-Notation

The Greek letter "∈" (epsilon) is used to denote that an object is an *element* of a set. When crossed out "∉" denotes that the object is *not an element*."

Ex.: $3 \in S$ reads:

"3 is an element of the set *S* ".

A set *S* is said to be a ***subset*** of the set *T* iff every element of *S* is also an element of *T*. This situation is denoted by

$S \subseteq T$

Specifying Sets

– By listing elements :

Ex.

- $S_1 = \{1, 2, 4, 5, 10, 20\}$
- $S_2 = \{0, 2, 4, \ldots\}$

– By defining property :

Ex.

- $S_1 = \{n : n \in N, \text{ n divides } 20\}$
- $S_2 = \{n : n \in N, \text{ n is even number}\}$

Examples

– { 11, 12, 13 }

– {  , ,  }

– {  , ,  , 11, Leo }

– { 11, 11, 11, 12, 13 } = { 11, 12, 13 } ?

– {  , ,  } = {  ,  ,  } ?

The Empty Set

The ***empty set*** is the set containing no elements. This set is also called the ***null set*** and is denoted by:

– { }

– ∅

Quiz

1. $\varnothing \subseteq \varnothing$? **(yes)**
2. $\varnothing \subset \varnothing$? **(No)**

Cardinality

The ***cardinality*** of a set is the number of distinct elements in the set. $|S|$ denotes the cardinality of S.

Q: Compute each cardinality.

1. $|\{1, -13, 4, -13, 1\}|$? (3)
2. $|\{3, \{1,2,3,4\}, \varnothing\}|$? (3)
3. $|\{\}|$? (0)
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}|$? (3)

Set Theoretic Operations

Set theoretic operations allow us to build new sets out of old.

Given sets A and B, the set theoretic operators are:

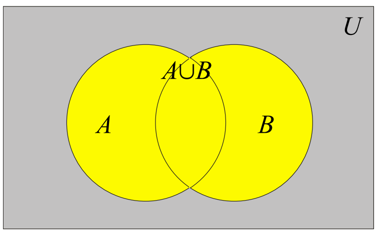
- Union ($\cup$)
- Intersection ($\cap$)
- Difference (-)
- Complement ("—")
- Cartesian Product: A×B
- Power set: P(A)

give us new sets $A\cup B$, $A\cap B$, A-B, $\overline{A}$, AxB, and P(A).

Union

Elements in at least one of the two sets:

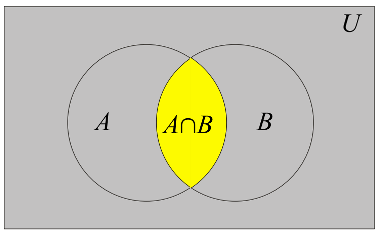
$$A\cup B = \{ x \mid x \in A \lor x \in B \}$$



Intersection

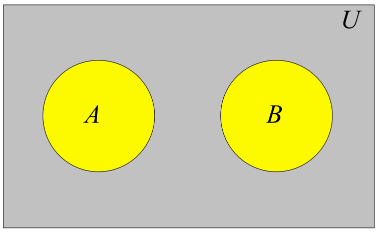
Elements in exactly one of the two sets:

$$A\cap B = \{ x \mid x \in A \land x \in B \}$$



Disjoint Sets

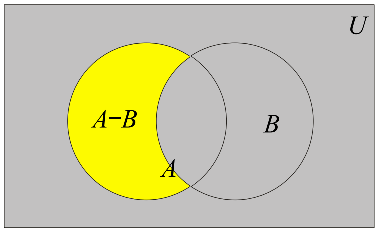
DEF: If A and B have no common elements, they are said to be ***disjoint***, i.e. $A \cap B = \varnothing$.



Set Difference

Elements in first set but not second:

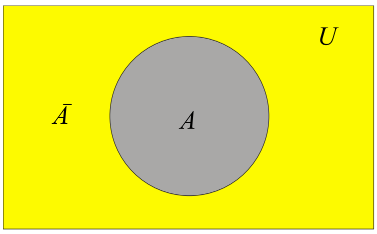
$A-B = \{ x \mid x \in A \wedge x \notin B \}$



Complement

Elements not in the set (unary operator):

$\bar{A} = \{ x \mid x \notin A \}$



Cartesian Product

The most famous example of 2-tuples are points in the Cartesian plane $\mathbf{R}^2$. Here ordered pairs (x,y) of elements of **R** describe the coordinates of each point. We can think of the first coordinate as the value on the x-axis and the second coordinate as the value on the y-axis.

The ***Cartesian product*** of two sets A and B (denoted by $A \times B$) is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

- $A \times B = \{ (x,y) \mid x \in A \text{ and } y \in B \}$

Q: What does $\varnothing \times S$ equal? (= ∅)

Some Examples

$L_{<6} = \{ x \mid x \in N, x<6 \}$
$L_{<6} \cap L_{prime} = \{2,3,5\}$

$\Sigma = \{0,1\}$
$\Sigma \times \Sigma = \{(0,0), (0,1), (1,0), (1,1)\}$

Power Sets

Formal: $P(A) = \{ S \mid S \subseteq A \}$

Example: $A = \{x,y\}$
$P(A) = \{ \{\} , \{x\} , \{y\} , \{x,y\} \}$

Note the different sizes:
$| P(A)| = 2^{|A|}$
$|A \times A| = |A|^2$

Power Sets

The ***power set*** of *S* is the set of all subsets of *S*.

Denote the power set by $P(S)$ or by 2^S.

The latter weird notation comes from the following lemma.

Lemma: $| 2^S | = 2^{|S|}$

Power Sets: Example

To understand the previous fact consider

$S = \{1,2,3\}$

Enumerate all the subsets of *S* :

0-element sets: {} 1

1-element sets: {1}, {2}, {3} + 3

2-element sets: {1,2}, {1,3}, {2,3} + 3

3-element sets: {1,2,3} + 1

Therefore: $| 2^S | = 8 = 2^3 = 2^{|S|}$

Binary Relations

A binary relation R is a set of pairs of elements of sets A and B, i.e. $R \subseteq A \times B$

- *A* is called the domain of R
- *B* is called the range (or codomain) of R
- If *A*=*B* we say that R is a relation on *A*
- We may write *aRb* for $(a,b) \in R$

Properties of Binary Relations

Special properties for relation *on* a set *A*:

- ***reflexive*** : every element is self-related. i.e. *aRa* for all $a \in A$
- ***symmetric*** : order is irrelevant. i.e. for all $a,b \in A$ *aRb* iff *bRa*
- ***transitive*** : when *a* is related to *b* and *b* is related to *c,* it follows that *a* is related to *c*. i.e. for all $a,b,c \in A$ *aRb* and *bRc* implies *aRc*

Properties of Binary Relations

- ***asymmetric*** : also *not* equivalent to "not symmetric". Meaning: it's never the case that both *aRb* and *bRa* hold.
  i.e. *aRb* ➔ $b \not R a$
- ***irreflexive*** : *not* equivalent to "not reflexive". Meaning: it's never the case that *aRa* holds.
  i.e. *for all a,* $a \not R a$

Properties of Binary Relations

An *equivalence relation* R is a relation on a set *A* which is reflexive, symmetric and transitive.

- **Generalizes the notion of "equals".**
- **R partitions A into disjoint nonempty equivalence classes, i.e., $A=A_1 \cup A_2 \cup \ldots$, such that**
  - **$A_i \cap A_j = \varnothing$, for all $i \neq j$**
  - **$a, b \in A_i$ ➔ aRb**
  - **$a \in A_i$, $b \in A_j$, $i \neq j$ ➔ $a \not R b$**
  - **A_i's are called equivalence classes and their number may be infinite**

Examples

- Set of Natural numbers is partitioned by the relation R={(i, j): i = j "mod 5"} into five "equivalence classes":
  { {0,5,10,…}, {1,6,11,…}, {2,7,12,…}, {3,8,13,…}, {4,9,14,…} }
- "String length" can be used to partition the set of all bit strings.
  { {},{0,1},{00,01,10,11},{000,…,111},… }

Closures of Relations

- If P is a set of properties of relations, the P-closure of a relation R is the smallest relation R' such that:
  1. R ⊆ R'
  2. aR' b ➔ P((a, b)) is true
  3. No more elements in R'

- Ex. The transitive-closure of a relation R, denoted by R^+ is defined by:
  1. aRb ➔ (a, b) ∈ R^+
  2. (a, b) ∈ R^+ , (b, c) ∈ R^+ ➔ (a, c) ∈ R^+
  3. Only elements in (1) and (2) are in R^+

- Note that: the {reflexive, transitive}-closure of a relation R is denoted by R*

Examples

- For the relation R = {(1,2), (2,2), (2,3)} on the set {1,2,3},
- R^+ and R^* are
- R^+ ={(1,2), (2,2), (2,3), (1, 3)} R + Transitive
- R^* ={(1,1), (1,2), (2,2), (2,3), (1, 3), (3,3)} R + Transitive + Reflexive