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Comments on "Energy-Efficient Beamforming Design for MU-MISO Mixed RF/VLC Heterogeneous Wireless Networks"

Thanh V. Pham *Member, IEEE,* and Anh T. Pham *Senior Member, IEEE*

***Abstract*—We show that the use of *Schur complement lemma* to derive equivalent convex constraints to those non-convex in (54) and (55) of the above paper is not valid. In this comment, an alternative approach is presented to convexify those constraints.**

***Index Terms*—Visible light communication, beamforming design, convex optimization, energy efficiency.**

I. Introduction

In [1], the authors studied coordinated beamforming designs for mixed RF/VLC heterogeneous networks from the perspective of energy efficiency. In particular for the case imperfect channel state information (CSI), a robust beamforming was formulated to take into account the channel estimation errors. The nature of this robust design essentially gave rise to infinitely many non-convex constraints, which render the optimal solution infeasible. To handle this, the *S-procedure* [2] and *Schur complement lemma* [2], [3] were used. Unfortunately, the use of *Schur complement lemma* as presented by the authors is not valid since a condition for the lemma being applicable does not hold. As a result, the derived convex constraints are not equivalent to the original non-convex ones. In this comment, we propose an alternative approach, which uses semidefinite relaxation (SDR) technique [4], to convexify the non-convex constraints.

II. The Schur Complement Lemma

Using the *S-procedure*, the infinitely many constraints [1, (54)] and [1, (55)] are reformulated into a finite number of positive-semidefinite constraints as

$$\begin{bmatrix} \lambda_k \mathbf{I}_N + \mathbf{w}_k\mathbf{w}_k^T & \mathbf{w}_k\mathbf{w}_k^T\hat{\mathbf{h}}_k \\ \hat{\mathbf{h}}_k^T\mathbf{w}_k\mathbf{w}_k^T & -\lambda_k\left(\epsilon_k^h\right)^2 + \hat{\mathbf{h}}_k^T\mathbf{w}_k\mathbf{w}_k^T\hat{\mathbf{h}}_k - \rho_k \end{bmatrix} \succeq \mathbf{0}, \tag{1}$$

$$\begin{bmatrix} \lambda_{k,k} \mathbf{I}_{N_t} + \mathbf{v}_k\mathbf{v}_k^H & \mathbf{v}_k\mathbf{v}_k^H\hat{\mathbf{g}}_k \\ \hat{\mathbf{g}}_k^H\mathbf{v}_k\mathbf{v}_k^H & -\lambda_{k,k}\left(\epsilon_k^g\right)^2 + \hat{\mathbf{g}}_k^H\mathbf{v}_k\mathbf{v}_k^H\hat{\mathbf{g}}_k - z_{k,k} \end{bmatrix} \succeq \mathbf{0}, \tag{2}$$

The above constraints are not convex due to quadratic forms in each matrix. The authors then made use of the *Schur complement lemma* to obtain the claimed equivalent convex constraints. The lemma cited from [3, **Lemma 18**] states that:

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Schur complement lemma [3]: Suppose $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are respectively $n \times n$, $n \times p$, $p \times n$ and $p \times p$ matrices, and $\mathbf{A}$ is invertible. Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{3}$$

so that $\mathbf{M}$ is a $(n+p)\times(n+p)$ matrix. The Schur complement of the block $\mathbf{D}$ of the matrix $\mathbf{M}$ is the $n \times n$ matrix

$$\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}. \tag{4}$$

Let $\mathbf{D}$ be positive definite. Then $\mathbf{M}$ is positive semi-definite if and only if the Schur complement of $\mathbf{D}$ in $\mathbf{M}$ is positive semidefinite.

The lemma can also be equivalently stated as in the authors' paper: The Schur complement of the block $\mathbf{A}$ of the matrix $\mathbf{M}$ is the $p \times p$ matrix

$$\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}. \tag{5}$$

Let $\mathbf{A}$ be positive definite. Then $\mathbf{M}$ is positive semi-definite if and only if the Schur complement of $\mathbf{A}$ in $\mathbf{M}$ is positive semidefinite.

Applying the above stated lemma to (1) with $\mathbf{A} = 1$, $\mathbf{B} = \begin{bmatrix} \mathbf{w}_k^T & \left(\hat{\mathbf{h}}_k^T\mathbf{w}_k\right)^T \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -\mathbf{w}_k \\ -\hat{\mathbf{h}}_k^T\mathbf{w}_k \end{bmatrix}$, and $\mathbf{D} = \begin{bmatrix} \lambda_k\mathbf{I}_N & 0 \\ 0 & -\rho_k - \lambda_k\left(\epsilon_k^h\right)^2 \end{bmatrix}$, we obtain the constraint [1, (59)] (similarly for constraints [1, (60)] and [1, (61)]). However, the above mentioned *Schur complement lemma* as the authors cited from [3] is not in its correct form. In fact, the correct statement of the lemma requires that $\mathbf{C}$ is the transpose of $\mathbf{B}$ [2, pp. 650]. Obviously, this is not always the case for constraints [1, (59)] and [1, (60)]. Hence, [1, (54)] and [1, (55)] are, in general, not equivalent to [1, (59)] and [1, (60)] as claimed by the authors. More importantly, the use of [1, (54)] and [1, (55)] would result in an infeasibility of the original problem [1, Problem (62)]. Indeed, it is obvious to see that the only possible values of $\mathbf{w}_k$ and $\mathbf{v}_k$ satisfying [1, (54)] and [1, (55)] respectively are $\mathbf{w}_k = \mathbf{0}$ and $\mathbf{v}_k = \mathbf{0}$. The resulting $R_k^{\text{VLC}}\{\mathbf{w}_k\}$ and $R_k^{\text{RF}}\{\mathbf{v}_k\}$ are thus 0 as well. Hence, the constraint [1, (13b)] can not be satisfied unless $\Gamma_k = 0$, which however makes the energy efficiency be 0.

III. Alternative Approach

In this section, we present an alternative approach using SDR technique. SDR works by introducing $\mathbf{W}_k = \mathbf{w}_k\mathbf{w}_k^T$

and $\mathbf{V}_k = \mathbf{v}_k\mathbf{v}_k^H$, $\forall k$. Accordingly, (1), (2), and [1, (56)] are equivalent to

$$\begin{bmatrix}\lambda_k\mathbf{I}_N + \mathbf{W}_k & \mathbf{W}_k\hat{\mathbf{h}}_k\\ \hat{\mathbf{h}}_k^T\mathbf{W}_k & -\lambda_k\left(\epsilon_k^h\right)^2 + \hat{\mathbf{h}}_k^T\mathbf{W}_k\hat{\mathbf{h}}_k - \rho_k\end{bmatrix} \succeq \mathbf{0},$$
$$\mathbf{W}_k \succeq \mathbf{0},\ \ \text{rank}\left(\mathbf{W}_k\right) = 1, \tag{6}$$

$$\begin{bmatrix}\lambda_{k,k}\mathbf{I}_{N_t} + \mathbf{V}_k & \mathbf{V}_k\hat{\mathbf{g}}_k\\ \hat{\mathbf{g}}_k^H\mathbf{V}_k & -\lambda_{k,k}\left(\epsilon_k^g\right)^2 + \hat{\mathbf{g}}_k^H\mathbf{V}_k\hat{\mathbf{g}}_k - z_{k,k}\end{bmatrix} \succeq \mathbf{0},$$
$$\mathbf{V}_k \succeq \mathbf{0}, \text{rank}\left(\mathbf{V}_k\right) = 1, \tag{7}$$

$$\begin{bmatrix}\lambda_{k,j}\mathbf{I}_{N_t} - \mathbf{V}_j & -\mathbf{V}_j\hat{\mathbf{g}}_k\\ -\hat{\mathbf{g}}_k^H\mathbf{V}_j & -\lambda_{k,j}\left(\epsilon_k^g\right)^2 - \hat{\mathbf{g}}_k^H\mathbf{V}_j\hat{\mathbf{g}}_k + z_{k,j}\end{bmatrix} \succeq \mathbf{0},$$
$$\mathbf{V}_j \succeq \mathbf{0},\ \ \text{rank}\left(\mathbf{V}_j\right) = 1. \tag{8}$$

In [1], two algorithms were proposed to solve the robust beamforming design, namely: *Robust Dinkelbach Algorithm Combined with Successive Convex Approximation (SCA)* and *Robust Low-Complexity SCA Algorithm*. For brevity, we describe the use of SDR for the former in the following. The use of SDR for the latter follows the same manner.

Similar to [1, Problem (62)], however with the use of (6), (7), and (8), the algorithm involves solving the following surrogate problem

$$\mathbf{Q}_{|(\mathbf{\Phi})} \triangleq \underset{\substack{\mathbf{W}_k,\mathbf{V}_k,r_k^{\text{VLC}},r_k^{\text{RF}}\\ \rho_k,z_{k,k},z_{k,j},\alpha_k,\beta_k\\ \lambda_k\geq 0,\lambda_{k,k}\geq 0,\lambda_{k,j}\geq 0}}{\text{maximize}} \left(\sum_{k=1}^{K} r_k^{\text{VLC}} + \sum_{k=1}^{K} r_k^{\text{RF}}\right) -$$
$$\eta\left(P_{\text{VLC}} + P_{\text{RF}} + \xi_1\sum_{k=1}^{K}\text{Tr}\left(\mathbf{W}_k\right) + \xi_2\sum_{k=1}^{K}\text{Tr}\left(\mathbf{V}_k\right)\right), \tag{9a}$$

$$\text{s.t. } r_{\text{VLC}}^k \leq \frac{1}{2}\log_2\left(1 + \frac{2\rho_k}{\pi e\sigma_{z_k}^2}\right),\ \ \forall k, \tag{9b}$$

$$r_{\text{RF}}^k \leq \log_2\left(1 + e^{\bar{\alpha}_k}\right) + \frac{e^{\bar{\alpha}_k}\left(\bar{\alpha}_k - \alpha_k\right)}{\ln 2\left(1 + e^{\bar{\alpha}_k}\right)},\ \ \forall k, \tag{9c}$$

$$e^{\alpha_k + \beta_k} \leq z_{k,k},\ \ \forall k, \tag{9d}$$

$$e^{\bar{\beta}_k}\left(\beta_k - \bar{\beta}_k + 1\right) \geq \sum_{j\neq k} z_{k,j} + \sigma_{n_k}^2,\ \ \forall k, \tag{9e}$$

$$\frac{\sum_{k=1}^{K}\mathbf{e}_n^T\mathbf{W}_k\mathbf{e}_n}{K} \leq \left(\min\{\text{p}_n, \text{p}_n - \text{p}_{max}\}\right)^2,\ \ \forall n, \tag{9f}$$

$$\sum_{k=1}^{K}\text{Tr}\left(\mathbf{V}_k\right) \leq P_{RF,max}, \tag{9g}$$

$$\hat{\mathbf{h}}_i^T\mathbf{W}_k\hat{\mathbf{h}}_i = 0,\ \ \forall i \neq k, \tag{9h}$$

$$r_{\text{VLC}}^k + r_{\text{RF}}^k \geq \Gamma_k,\ \ \forall k, \tag{9i}$$

$$\begin{bmatrix}\lambda_k\mathbf{I}_N + \mathbf{W}_k & \mathbf{W}_k\hat{\mathbf{h}}_k\\ \hat{\mathbf{h}}_k^T\mathbf{W}_k & -\lambda_k\left(\epsilon_k^h\right)^2 + \hat{\mathbf{h}}_k^T\mathbf{W}_k\hat{\mathbf{h}}_k - \rho_k\end{bmatrix} \succeq \mathbf{0}, \forall k, \tag{9j}$$

$$\begin{bmatrix}\lambda_{k,k}\mathbf{I}_{N_t} + \mathbf{V}_k & \mathbf{V}_k\hat{\mathbf{g}}_k\\ \hat{\mathbf{g}}_k^H\mathbf{V}_k & -\lambda_{k,k}\left(\epsilon_k^g\right)^2 + \hat{\mathbf{g}}_k^H\mathbf{V}_k\hat{\mathbf{g}}_k - z_{k,k}\end{bmatrix} \succeq \mathbf{0}, \forall k, \tag{9k}$$

$$\begin{bmatrix}\lambda_{k,j}\mathbf{I}_{N_t} - \mathbf{V}_j & -\mathbf{V}_j\hat{\mathbf{g}}_k\\ -\hat{\mathbf{g}}_k^H\mathbf{V}_j & -\lambda_{k,j}\left(\epsilon_k^g\right)^2 - \hat{\mathbf{g}}_k^H\mathbf{V}_j\hat{\mathbf{g}}_k + z_{k,j}\end{bmatrix} \succeq \mathbf{0},\ \ \forall j \neq k, \tag{9l}$$

$$\mathbf{W}_k \succeq \mathbf{0},\ \ \forall k, \tag{9m}$$

$$\mathbf{V}_k \succeq \mathbf{0},\ \ \forall k, \tag{9n}$$

$$\text{rank}\left(\mathbf{W}_k\right) = 1,\ \ \forall k, \tag{9o}$$

$$\text{rank}\left(\mathbf{V}_k\right) = 1,\ \ \forall k. \tag{9p}$$

It should be noted that the original constraint [1, (13d)] can not be equivalently represented in terms of $\mathbf{W}_k$. Instead, due to $\frac{\left(\sum_{k=1}^{K}\left|\mathbf{e}_n^T\mathbf{w}_k\right|\right)^2}{K} \leq \sum_{k=1}^{K}\mathbf{e}_n^T\mathbf{W}_k\mathbf{e}_n$, we replace it by a more stringent constraint (9f). As a result, problem (9) generally gives a lower bound solution to the original problem [1, Problem(43)]. It is seen that except (9o) and (9p), all constraints (9b)-(9n) are convex. Thus, we omit (9o), (9p) and solve the following convex optimization problem

$$\mathbf{Q}_{1|(\mathbf{\Phi})} \triangleq \underset{\substack{\mathbf{W}_k,\mathbf{V}_k,r_k^{\text{VLC}},r_k^{\text{RF}}\\ \rho_k,z_{k,k},z_{k,j},\alpha_k,\beta_k\\ \lambda_k\geq 0,\lambda_{k,k}\geq 0,\lambda_{k,j}\geq 0}}{\text{maximize}} \left(\sum_{k=1}^{K} r_k^{\text{VLC}} + \sum_{k=1}^{K} r_k^{\text{RF}}\right) -$$
$$\eta\left(P_{\text{VLC}} + P_{\text{RF}} + \xi_1\sum_{k=1}^{K}\text{Tr}\left(\mathbf{W}_k\right) + \xi_2\sum_{k=1}^{K}\text{Tr}\left(\mathbf{V}_k\right)\right), \tag{10}$$
$$\text{s.t. (9b)} - \text{(9n)}.$$

The remaining issue now is to verify whether the solution to (10) is also optimal for (9). Indeed, the following theorem proves the equivalence of the two problems.

Theorem 1: If problem (10) is feasible then its optimal solutions $\mathbf{W}_k^*$ and $\mathbf{V}_k^*$ always satisfy that $\text{rank}\left(\mathbf{W}_k^*\right) = 1$ and $\text{rank}\left(\mathbf{V}_k^*\right) = 1$.

For brevity, we omit the proof of this theorem in this comment. Interested readers may find it in [5]. In addition, one can realize that an upper bound solution to [1, Problem(43)] can be given by replacing [1, (13d)] by a looser constraint as $\sum_{k=1}^{K}\mathbf{e}_n^T\mathbf{W}_k\mathbf{e}_n \leq \left(\sum_{k=1}^{K}\left|\mathbf{e}_n^T\mathbf{w}_k\right|\right)^2$. This constraint replacement, however, does not change the conclusion of **Theorem 1**.

IV. Conclusion

In this comment, we showed that the use of *Schur complement lemma* to derive equivalent convex constraints to [1, (54)] and [1, (55)] is not valid. While the two proposed algorithms (i.e. [1, **Algorithm 4** and **Algorithm 5**]) is still applicable in the sense that their convergence properties are unchanged, without the mentioned equivalence, the numerical results related to the robust beamforming design (i.e. Figs. 7, 8, 9, and 10) may not be valid [5].

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A Proof of the Equivalence of Semidefinite Relaxation

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***Abstract*—In this note, we present a proof of Theorem 1 and numerical examples to demonstrate the feasibility of our proposed approach in [1].**

I. Proof of Theorem 1 [1]

Let us recall [1, Problem (10)] written as follows

$$\mathbf{Q}_{|(\mathbf{\Phi})} \triangleq \underset{\substack{\mathbf{W}_k,\mathbf{V}_k,r_k^{\text{VLC}},r_k^{\text{RF}} \\ \rho_k,z_{k,k},z_{k,j},\alpha_k,\beta_k \\ \lambda_k\geq 0,\lambda_{k,k}\geq 0,\lambda_{k,j}\geq 0}}{\text{maximize}} \left(\sum_{k=1}^{K} r_k^{\text{VLC}} + \sum_{k=1}^{K} r_k^{\text{RF}}\right) - \eta\left(P_{\text{VLC}} + P_{\text{RF}} + \xi_1 \sum_{k=1}^{K}\text{Tr}\left(\mathbf{W}_k\right) + \xi_2 \sum_{k=1}^{K}\text{Tr}\left(\mathbf{V}_k\right)\right), \quad \text{(1a)}$$

$$\text{s.t. } r_{\text{VLC}}^k \leq \frac{1}{2}\log_2\left(1 + \frac{2\rho_k}{\pi e \sigma_{z_k}^2}\right), \ \forall k, \quad \text{(1b)}$$

$$r_{\text{RF}}^k \leq \log_2\left(1 + e^{\bar{\alpha}_k}\right) + \frac{e^{\bar{\alpha}_k}\left(\bar{\alpha}_k - \alpha_k\right)}{\ln 2\left(1 + e^{\bar{\alpha}_k}\right)}, \ \forall k, \quad \text{(1c)}$$

$$e^{\alpha_k + \beta_k} \leq z_{k,k}, \ \forall k, \quad \text{(1d)}$$

$$e^{\bar{\beta}_k}\left(\beta_k - \bar{\beta}_k + 1\right) \geq \sum_{j\neq k} z_{k,j} + \sigma_{n_k}^2, \ \forall k, \quad \text{(1e)}$$

$$\frac{\sum_{k=1}^{K}\mathbf{e}_n^T\mathbf{W}_k\mathbf{e}_n}{K} \leq \left(\min\{\text{p}_n, \text{p}_n - \text{p}_{max}\}\right)^2, \ \forall n, \quad \text{(1f)}$$

$$\sum_{k=1}^{K}\text{Tr}\left(\mathbf{V}_k\right) \leq P_{RF,max}, \quad \text{(1g)}$$

$$\hat{\mathbf{h}}_i^T\mathbf{W}_k\hat{\mathbf{h}}_i = 0, \ \forall i \neq k, \quad \text{(1h)}$$

$$r_{\text{VLC}}^k + r_{\text{RF}}^k \geq \Gamma_k, \ \forall k, \quad \text{(1i)}$$

$$\begin{bmatrix} \lambda_k\mathbf{I}_N + \mathbf{W}_k & \mathbf{W}_k\hat{\mathbf{h}}_k \\ \hat{\mathbf{h}}_k^T\mathbf{W}_k & -\lambda_k\left(\epsilon_k^h\right)^2 + \hat{\mathbf{h}}_k^T\mathbf{W}_k\hat{\mathbf{h}}_k - \rho_k \end{bmatrix} \succeq \mathbf{0}, \forall k, \quad \text{(1j)}$$

$$\begin{bmatrix} \lambda_{k,k}\mathbf{I}_{N_t} + \mathbf{V}_k & \mathbf{V}_k\hat{\mathbf{g}}_k \\ \hat{\mathbf{g}}_k^H\mathbf{V}_k & -\lambda_{k,k}\left(\epsilon_k^g\right)^2 + \hat{\mathbf{g}}_k^H\mathbf{V}_k\hat{\mathbf{g}}_k - z_{k,k} \end{bmatrix} \succeq \mathbf{0}, \forall k, \quad \text{(1k)}$$

$$\begin{bmatrix} \lambda_{k,j}\mathbf{I}_{N_t} - \mathbf{V}_j & -\mathbf{V}_j\hat{\mathbf{g}}_k \\ -\hat{\mathbf{g}}_k^H\mathbf{V}_j & -\lambda_{k,j}\left(\epsilon_k^g\right)^2 - \hat{\mathbf{g}}_k^H\mathbf{V}_j\hat{\mathbf{g}}_k + z_{k,j} \end{bmatrix} \succeq \mathbf{0}, \ \forall j \neq k, \quad \text{(1l)}$$

$$\mathbf{W}_k \succeq \mathbf{0}, \ \forall k, \quad \text{(1m)}$$

$$\mathbf{V}_k \succeq \mathbf{0}, \ \forall k, \quad \text{(1n)}$$

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We first proof that the optimal $\mathbf{W}_k^*$ satisfies rank $(\mathbf{W}^*) = 1$. The Karush-Kuhn-Tucker (KKT) equations relevant to the optimal $\mathbf{W}_k^*$ are given by

$$\mathbf{X}_k^{4^*} + \hat{\mathbf{H}}_k\mathbf{X}_k^{1^*}\hat{\mathbf{H}}_k^T = \xi_1\mathbf{I}_N + \sum_{n=1}^{N}\varphi_n^*\mathbf{e}_n\mathbf{e}_n^T + \sum_{i=1,i\neq k}^{K}\omega_i^*\hat{\mathbf{h}}_i\hat{\mathbf{h}}_i^T, \ \forall k, \quad (2)$$

$$\mathbf{X}_k^{1^*}\left(\begin{bmatrix} \lambda_k^*\mathbf{I}_N & \mathbf{0}_N \\ \mathbf{0}_N^T & -\lambda_k^*\left(\epsilon_k^h\right)^2 - \rho_k^* \end{bmatrix} + \hat{\mathbf{H}}_k^T\mathbf{W}_k^*\hat{\mathbf{H}}_k\right) = \mathbf{0}, \ \forall k, \quad (3)$$

$$\mathbf{X}_k^{4^*}\mathbf{W}_k^* = \mathbf{0}, \ \forall k, \quad (4)$$

where $\{\varphi_n^*\}$, $\{\omega_n^*\}$, $\mathbf{X}_k^{1^*}$, $\mathbf{X}_k^{4^*}$ are the optimal solutions to the their respective dual variables (i.e. $\{\varphi_n\} \geq 0$, $\{\omega_i\} \geq 0$, $\mathbf{X}_k^1 \succeq \mathbf{0}$, $\mathbf{X}_k^4 \succeq \mathbf{0}$) associated with constraints (1f), (1h), (1j), and (1m). At the optimal solution, let $\Delta\mathbf{h}_k^*$ be the error vector that $\left(\hat{\mathbf{h}}_k^T + \Delta\mathbf{h}_k^{*T}\right)\mathbf{W}_k^*\left(\hat{\mathbf{h}}_k + \Delta\mathbf{h}_k^*\right) = \rho_k^* = \min_{\|\Delta\mathbf{h}\|^2\leq\left(\epsilon_k^h\right)^2}\left(\hat{\mathbf{h}}_0^T + \Delta\mathbf{h}_k^{*T}\right)\mathbf{V}_{\text{rb}}^*\left(\hat{\mathbf{h}}_0 + \Delta\mathbf{h}_k^*\right)$ and let $\mathbf{A}_k = \begin{bmatrix} \lambda_k^*\mathbf{I}_N + \mathbf{W}_k^* & \mathbf{W}_k^*\hat{\mathbf{h}}_k \\ \hat{\mathbf{h}}_k^T\mathbf{W}_k^* & -\lambda_k^*\left(\epsilon_k^h\right)^2 + \hat{\mathbf{h}}_k^T\mathbf{W}_k^*\hat{\mathbf{h}}_k - \rho_k^*. \end{bmatrix}$ The fact that $\mathbf{A}_k \succeq \mathbf{0}$ implies

$$\begin{aligned} &\begin{bmatrix}\Delta\mathbf{h}_k^{*T} & 1\end{bmatrix}\mathbf{A}_k\begin{bmatrix}\Delta\mathbf{h}_k^* \\ 1\end{bmatrix} \\ &= \left(\left(\hat{\mathbf{h}}_k^T + \Delta\mathbf{h}_k^{*T}\right)\mathbf{W}_k^*\left(\hat{\mathbf{h}}_k + \Delta\mathbf{h}_k^*\right) - \rho_k^*\right) \\ &+ \lambda_k^*\left(\|\Delta\mathbf{h}_k^*\|_2^2 - \left(\epsilon_k^h\right)^2\right) = \lambda_k^*\left(\|\Delta\mathbf{h}_k^*\|^2 - \left(\epsilon_k^h\right)^2\right) \geq 0. \end{aligned} \quad (5)$$

Since $\|\Delta\mathbf{h}_k^*\|^2 \leq \left(\epsilon_k^h\right)^2$ and $\lambda_k^* \geq 0$, we conclude that $\lambda_k^*\left(\|\Delta\mathbf{h}_k^*\|^2 - \left(\epsilon_k^h\right)^2\right) = 0$. As a result, we get $\begin{bmatrix}\Delta\mathbf{h}_k^{*T} & 1\end{bmatrix}\mathbf{A}_k = \mathbf{0}$. This implies that

$$\text{rank}\left(\mathbf{A}_k\right) = \text{rank}\left(\begin{bmatrix}\lambda_k^*\mathbf{I}_N + \mathbf{W}_k^* & \mathbf{W}_k^*\hat{\mathbf{h}}_k\end{bmatrix}\right), \quad (6)$$

because $\begin{bmatrix}\lambda_k^*\mathbf{I}_N + \mathbf{W}_k^* & \mathbf{W}^*{}_k\hat{\mathbf{h}}_k\end{bmatrix}$ is a linear combination of the row vectors of $\begin{bmatrix}\lambda_k^*\mathbf{I}_N + \mathbf{W}_k^* & \mathbf{W}_k^*\hat{\mathbf{h}}_k\end{bmatrix}$. Similarly, since $\mathbf{A}_k\begin{bmatrix}\Delta\mathbf{h}_k^* \\ 1\end{bmatrix} = \mathbf{0}$, $\mathbf{W}_k^*\hat{\mathbf{h}}_k$ can also be linearly expressed by the column vectors of $[\lambda_k^*\mathbf{I}_N + \mathbf{W}_k^*]$. Together with (6), we get

$$\text{rank}\left(\mathbf{A}_k\right) = \text{rank}\left(\left[\lambda_k^*\mathbf{I}_N + \mathbf{W}_k^*\right]\right). \quad (7)$$

If $\lambda_k^* = 0$ then

$$\begin{bmatrix}\Delta\mathbf{h}_k^{*T} & 1\end{bmatrix}\begin{bmatrix} \mathbf{W}_k^* & \mathbf{W}_k^*\hat{\mathbf{h}}_k \\ \hat{\mathbf{h}}_k^T\mathbf{W}_k^* & \hat{\mathbf{h}}_k^T\mathbf{W}_k^*\hat{\mathbf{h}}_k - \rho_k^* \end{bmatrix}\begin{bmatrix}\Delta\mathbf{h}_k^* \\ 1\end{bmatrix} = 0 \quad (8)$$

No.		1	2	3	4	5
VLC channel		$10^{-4}\begin{bmatrix}0.2014\\0.4076\\0.2050\\0.1214\end{bmatrix}$	$10^{-4}\begin{bmatrix}0.0997\\0.0566\\0.1263\\0.3160\end{bmatrix}$	$10^{-4}\begin{bmatrix}0.3815\\0.2594\\0.1255\\0.1627\end{bmatrix}$	$10^{-4}\begin{bmatrix}0.0730\\0.1800\\0.3393\\0.1068\end{bmatrix}$	$10^{-4}\begin{bmatrix}0.3619\\0.1531\\0.1303\\0.2839\end{bmatrix}$
RF channel		$\begin{bmatrix}0.3301+0.0256i\\-0.0339+0.4130i\\-0.0976+0.7635i\\-0.1088+0.2335i\\-0.1516-0.1049i\\0.0115+0.3126i\end{bmatrix}$	$\begin{bmatrix}0.1959+0.0062i\\-0.6253-1.5146i\\-0.4740-0.2285i\\-0.3706+0.6212i\\-0.2539-0.5334i\\-0.1603+0.4669i\end{bmatrix}$	$\begin{bmatrix}0.0912-0.3671i\\-0.7825-0.0154i\\-0.0423+0.1162i\\0.8020+0.2132i\\0.0492-0.1864i\\0.0207-0.1182i\end{bmatrix}$	$\begin{bmatrix}1.1147+0.2114i\\0.1688-0.8351i\\0.5000+0.2358i\\-0.8321-0.6064i\\-0.2950+0.0331i\\-0.1390+0.3262i\end{bmatrix}$	$\begin{bmatrix}-0.3452+0.5102i\\-0.3258+0.4309i\\0.5961+0.0006i\\-0.8059-0.0354i\\-0.0122-1.2431i\\-0.9744+0.2906i\end{bmatrix}$
Results						
Authors' method [2]		Infeasible	Infeasible	Infeasible	Infeasible	Infeasible
Our method	$\mathbf{w}_k$	$\begin{bmatrix}0.05\\0.05\\0.05\\0.05\end{bmatrix}$	$\begin{bmatrix}0.05\\0.05\\0.05\\0.05\end{bmatrix}$	$\begin{bmatrix}0.05\\0.05\\0.05\\0.05\end{bmatrix}$	$\begin{bmatrix}0.05\\0.05\\0.05\\0.05\end{bmatrix}$	$\begin{bmatrix}0.05\\0.05\\0.05\\0.05\end{bmatrix}$
	$\mathbf{v}_k$	$\begin{bmatrix}0.2747\\0.0040\\-0.0215\\-0.0718\\-0.1334\\0.0337\end{bmatrix}$	$\begin{bmatrix}-0.0769\\0.0831\\0.1628\\0.2141\\0.0426\\0.1141\end{bmatrix}$	$\begin{bmatrix}0.0846\\-0.2239\\-0.0307\\0.1981\\0.0438\\0.0247\end{bmatrix}$	$\begin{bmatrix}-0.2261\\-0.1357\\-0.0849\\0.1163\\0.0702\\0.0695\end{bmatrix}$	$\begin{bmatrix}0.0463\\0.0467\\-0.1323\\0.1809\\0.0765\\0.1990\end{bmatrix}$
	r_k^{VLC}	4.2903	3.6479	4.2807	3.8710	4.2810
	r_k^{RF}	17.0818	19.2098	20.1566	20.8746	20.9117
	η	2.7743×10^7	2.8655×10^7	3.1047×10^7	3.0937×10^7	3.1863×10^7

Table I: Numerical Results.

which implies $\mathbf{W}_k^*\left(\hat{\mathbf{h}}_k+\Delta\mathbf{h}_k^*\right)=\mathbf{0}$, or, in other words $\left(\hat{\mathbf{h}}_k^T+\Delta\mathbf{h}_k^{*T}\right)\mathbf{V}_{\text{rb}}^*\left(\hat{\mathbf{h}}_k+\Delta\mathbf{h}_k^*\right)=0$. As a result, $\left[\Delta\mathbf{h}_k^{*T}\quad 1\right]\begin{bmatrix}\mathbf{W}_k^* & \mathbf{W}_k^*\hat{\mathbf{h}}_k\\ \hat{\mathbf{h}}_k^T\mathbf{W}_k^* & \hat{\mathbf{h}}_k^T\mathbf{W}_k^*\hat{\mathbf{h}}_k-\rho_k^*\end{bmatrix}\begin{bmatrix}\Delta\mathbf{h}_k^*\\ 1\end{bmatrix}=\left(\hat{\mathbf{h}}_k^T+\Delta\mathbf{h}_k^{*T}\right)\mathbf{W}_k^*\left(\hat{\mathbf{h}}_k+\Delta\mathbf{h}_k^*\right)-\rho_0^*=-\rho_k^*$. This means $\rho_k^*=0$, which obviously violates the constraint in (1b) as a positive VLC channel rate is assumed. Therefore, $\lambda_k^*>0$, which results in $\text{rank}\left(\mathbf{A}_k\right)=N$.

From (3), according to the rank-nullity theorem

$$\text{rank}\left(\mathbf{X}_k^{1*}\right)\leq\text{Nullity}\left(\mathbf{A}_k\right)=(N+1)-\text{rank}\left(\mathbf{A}_k\right)=1. \tag{9}$$

From (2), since $\{\varphi_n^*\}\geq 0$, $\{\omega_i^*\}\geq 0$, and $\xi_1>0$ we have

$$\begin{aligned}&\text{rank}\left(\mathbf{X}_k^{4*}+\widehat{\mathbf{H}}_k\mathbf{X}_k^{1*}\widehat{\mathbf{H}}_k^T\right)\\&=\text{rank}\left(\xi_1\mathbf{I}_N+\sum_{n=1}^{N}\varphi_n^*\mathbf{e}_n\mathbf{e}_n^T+\sum_{i=1,i\neq k}^{K}\omega_i^*\hat{\mathbf{h}}_i\hat{\mathbf{h}}_i^T\right)\\&=N.\end{aligned} \tag{10}$$

Also, $\text{rank}\left(\mathbf{X}_k^{4*}+\widehat{\mathbf{H}}_k\mathbf{X}_k^{1*}\widehat{\mathbf{H}}_k^T\right)\leq\text{rank}\left(\mathbf{X}_k^{4*}\right)+\text{rank}\left(\widehat{\mathbf{H}}_k\mathbf{X}_k^{1*}\widehat{\mathbf{H}}_k^T\right)\leq\text{rank}\left(\mathbf{X}_k^{4*}\right)+1$. Thus, $\text{rank}\left(\mathbf{X}_k^{4*}\right)\geq N-1$. Moreover, we can see from (4) that $\text{rank}\left(\mathbf{W}_k^*\right)\leq\text{Nullity}\left(\mathbf{X}_k^{4*}\right)=N-\text{rank}\left(\mathbf{X}_k^{4*}\right)\leq 1$. If $\text{rank}\left(\mathbf{W}_k^*\right)=0$ then $\mathbf{W}_k^*=\mathbf{0}$, hence $\rho_k^*=0$, which does not satisfy the constraint in (1b). Therefore $\text{rank}\left(\mathbf{W}_k^*\right)=1$.

Following the same arguments, one can prove that the optimal $\mathbf{V}_k^*$ always satisfies $\text{rank}\left(\mathbf{V}_k^*\right)=1$. This completes the proof. The beamformers $\mathbf{w}_k$ and $\mathbf{v}_k$ are then obtained by $\mathbf{w}_k=\sqrt{\psi_w}\mathbf{q}_w$ and $\mathbf{v}_k=\sqrt{\psi_v}\mathbf{q}_v$, where $\mathbf{q}_w$ and $\mathbf{q}_v$ are the eigenvectors of $\mathbf{W}_k^*$ and $\mathbf{V}_k^*$, which associate with the non-zero eigenvalues ψ_w and ψ_v.

II. Numerical Examples

In this section, numerical results are presented to demonstrate the feasibility of our proposed approach. All parameters are the same as in [2]. As the authors did not mention the noise power of the RF link, we choose the typical value $\sigma_{n_1}^2=\cdots=\sigma_{n_K}^2=10^{-7}$, which corresponds to the noise spectral density $N_0=10^{-14}$. For brevity, simulations are performed for the *Robust Dinkelbach Algorithm Combined with SCA* (i.e. [2, **Algorithm 4**]) with 1 user ($K=1$) and 5 different random realizations of VLC and RF channels given in Table I. Initial points $\bar{\alpha}_k=\bar{\beta}_k=-1.5$, threshold $\Gamma_k=2$, maximum iteration number $L_{max}=50$, and error tolerance $\epsilon_4=10^{-3}$ are chosen. Numerical results are obtained using CVX [3] with SDPT3 solver version 4.0 and given in Table I. As expected, the authors' approach results in an infeasible design problem for all examined channel realizations while our approach gives explicit results of the beamformers $\mathbf{w}_k$ and $\mathbf{v}_k$.

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