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PAPER *Special Section on Information Theory and Its Applications*

A Novel Class of Structured Zero-Correlation Zone Sequence Sets

Takafumi HAYASHI[†a)], ***Senior Member***, **Takao MAEDA**[††b)], **Anh T. PHAM**[††c)], ***and*** **Shinya MATSUFUJI**[†††d)], ***Members***

SUMMARY The present paper introduces a novel type of structured ternary sequences having a zero-correlation zone (zcz) for both periodic and aperiodic correlation functions. The cross-correlation function and the side lobe of the auto-correlation function of the proposed sequence set are zero for phase shifts within the zcz. The proposed zcz sequence set can be generated from an arbitrary pair of an Hadamard matrix of order ℓ_h and a binary/ternary perfect sequence of length ℓ_p. The sequence set of order 0 is identical to the r-th row of the Hadamard matrix. For $m \geq 0$, the sequence set of order $(m+1)$ is constructed from the sequence set of order m by sequence concatenation and interleaving. The sequence set has ℓ_p subsets of size $2\ell_h$. The periodic correlation function and the aperiodic correlation function of the proposed sequence set have a zcz from $-(2^{m+1}-1)$ to $2^{m+1}-1$. The periodic correlation function and the aperiodic correlation function of the sequences of the i-th subset and k-th subset have a zcz from $-2^{m+2}(\ell_h+1)((j-k) \bmod \ell_p)$ to $-2^{m+2}(\ell_h+1)((j-k) \bmod \ell_p)$. The proposed sequence is suitable for a heterogeneous wireless network, which is one of the candidates for the fifth-generation mobile networks.

key words: *zero-correlation zone, ternary sequence, asymmetric* zcz*, inter-subset* zcz

1. Introduction

A sequence set having the property that the out-of-phase autocorrelation and cross-correlation functions are all equal to zero in a specified zone of phase shift is called a *zero-correlation zone* (zcz) sequence set [6]. Sequences having good correlation properties, such as zcz sequence sets, perfect sequences, complementary sequences, and M-sequences, are used in various applications, including communication systems, radars, position sensing, and ultrasonic imaging [3], [28], [40], [59].

We have proposed a class of zcz sequence sets, which are sometimes referred to as asymmetric zcz sequence sets, having subsets and a *wider inter-subset* zcz as follows [16], [21]: The correlation function of the sequences of a pair of different subsets, referred to as the *inter-subset correlation function*, has a zcz with a width wider than that of the correlation function of sequences of the same subset (*intra-subset correlation function*). There exist several sequence sets having similar subset and inter-subset correlation functions, which are wider than intra-subset correlation functions [13], [16], [18]–[21], [26], [35], [43], [44], [47], [48], [52], [55]–[58], [61]. We have proposed a class of ternary zcz sequence sets for both periodic and aperiodic correlation functions whose sequence sets can work with both even and odd correlations for spectrum spreading communications. [13], [16], [18]–[21].

The previously reported sequence set in [16] has simple subsets. However, we need a more complex structure of subsets for some applications. The proposed sequence construction scheme is an enhancement of previously reported schemes [13], [16], [18]–[20]. In this paper, we introduce a novel type of structure of the subsets.

For a given Hadamard matrix of order ℓ_h and a binary or ternary perfect sequence of length ℓ_p, the proposed sequence set of order m has ℓ_p subsets of size $4\ell_h$ for all $m \geq 0$. The length of the sequence is equal to $2^{m+2}(\ell_h+1)\ell_p$; The phase shift of the zcz for the whole sequence set is from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$.

The width of the zcz between a sequence of the j-th subset and one of the k-th subset is almost proportional to the difference $|j-k|$, which is defined as *the distance between the j-th subset and the k-th subset*. The structure of the proposed intra-subset zcz is quite different from previously reported structures [13], [16], [18]–[20], [26], [35], [47], [57], [61].

In this paper, we refer to a zcz sequence set which has a particular structure for the width of the zcz between a sequence of the j-th subset and one of the k-th subset as a structured zcz sequence set.

The wide intra-subset zcz of the proposed sequence set can achieve improved performance in applications of zcz sequence sets. For the application of a zcz sequence set to wireless communication systems, the zcz is mainly used for synchronization and channel separation. A zcz allows the signal time-delay within it to be canceled. Since the signal delay is longer for a longer signal path, the signal delay between a pair of devices of different cells is longer than the delay between a pair of devices of a common cell. Thus, the wide intra-subset zcz of the proposed sequence set can cancel a longer signal delay between different cell devices. For wireless communication applications, we must consider the distance between the particular cells. We need

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[†]The author is with Niigata University, Niigata-shi, 950-2181 Japan.
[††]The authors are with University of Aizu, Aizu-Wakamatsu-shi, 965-8580 Japan.
[†††]The author is with Yamaguchi University, Ube-shi, 755-8611 Japan.
a) E-mail: takafumi@ie.niigata-u.ac.jp
b) E-mail: t-maeda@u-aizu.ac.jp
c) E-mail: pham@u-aizu.ac.jp
d) E-mail: s-matsu@yamaguchi-u.ac.jp

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to prepare a longer ZCZ for a cell-pair which has a longer distance, which the existing ZCZ sequences are unable to achieve. We can assign subsets to cells provided that the path lengths of the nodes related to the subsets are longer for the cells which are farther apart from each other. Then, the proposed sequence set can realize a longer ZCZ for a greater distance between cells.

Applications of a ZCZ sequence set to CDMA systems have been widely investigated. In a DS-CDMA system, a sequence set is used for channel separation. However, inaccurate synchronization and multi-path propagation cause a time delay that destroys the orthogonality of the channel separation. In an approximately synchronized CDMA (AS-CDMA) system or quasi-synchronous CDMA (QS-CDMA) system, a ZCZ sequence set allows co-channel interference to be eliminated [40]. The application of a ternary ZCZ sequence set to an AS-CDMA system was demonstrated in [41]. The proposed ternary ZCZ sequence set can be applied to AS-CDMA in the same manner. The time delay is shorter for the nodes closer to the access point. In the case of the signals from a node far from the access point causing a larger time delay, the node is usually located in the cell of a different access point. A subset of the proposed tree-structured sequence set can be assigned to a cell. As mentioned above, for the cells which are distant from each other, assigning subsets to distant nodes should be performed according to the path lengths of the nodes.

In this paper, after an examination of preliminary considerations in Sect. 2, a scheme for constructing the proposed sequence set is presented in Sect. 3. The properties of the proposed sequence sets are shown in Sect. 4. The applications of the proposed sequence sets are discussed in Sect. 5. Finally, we present some concluding remarks in Sect. 6.

2. Preliminary Considerations

The following is a brief introduction of the definitions and notation used in the paper.

The floor function of x, $\lfloor x \rfloor$, gives the greatest integer that is less than or equal to x; the ceiling function of x, $\lceil x \rceil$, gives the least integer that is greater than or equal to x.

The quotient and modulo operations for integers a and b are denoted by $a \oslash b$ and $a \bmod b$ for $b > 0$, respectively, and are defined as follows:

$$a \oslash b = \begin{cases} \left\lfloor \frac{a}{b} \right\rfloor & \text{if } b > 0, \\ \left\lceil \frac{a}{b} \right\rceil & \text{if } b < 0, \end{cases} \tag{1a}$$

$$a \bmod b = a - b(a \oslash b). \tag{1b}$$

A set of k sequences $\boldsymbol{v}_r$ of length ℓ_v, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\}$, is denoted by $\{\boldsymbol{v}_r | 0 \leq r < k\}$, where $\boldsymbol{v}_r = [v_{r,0}, v_{r,1}, \ldots v_{r,\ell_v-1}]$.

The norm of a sequence $\boldsymbol{v}_r$ of length ℓ_v is denoted by $|\boldsymbol{v}_r|$ and is defined as $\sum_{j=0}^{\ell_v-1} v_{r,j}\, v_{r,j}$.

The aperiodic correlation function $\overset{\text{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau)$, the periodic correlation function $\overset{\text{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau)$, and the odd correlation function $\overset{\text{O}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau)$ of a pair of sequences $\boldsymbol{v}_r$ and $\boldsymbol{v}_s$ of length ℓ_v for a phase shift τ are defined, respectively, as follows:

$$\overset{\text{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau) = \begin{cases} \sum_{j=0}^{\ell_v-\tau-1} v_{r,j}\; v_{s,j+\tau} & \text{if } 0 \leq \tau < \ell_v, \\ \sum_{j=0}^{\ell_v+\tau-1} v_{r,j-\tau}\; v_{s,j} & \text{if } -\ell_v < \tau < 0, \\ 0 & \text{if } |\tau| \geq \ell_v; \end{cases} \tag{2a}$$

$\forall k, \quad 0 \leq \tau < \ell_v,$

$$\overset{\text{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(k\,\ell_v + \tau) = \overset{\text{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau) + \overset{\text{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau - \ell_v), \quad \text{and} \tag{2b}$$

$$\overset{\text{O}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(k\,\ell_v + \tau) = \overset{\text{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau) - \overset{\text{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau - \ell_v). \tag{2c}$$

When each of sequences $\boldsymbol{r}$ and $\boldsymbol{s}$ has a run of zero elements of length ℓ_0 at its tail, the correlation functions of the sequence pair $\boldsymbol{r}$ and $\boldsymbol{s}$, each of length ℓ_v, satisfy the following: If the absolute value of phase shift $|\tau|$ is less than or equal to the run length ℓ_0, then the aperiodic correlation function $\overset{\text{A}}{\theta}_{\boldsymbol{r},\boldsymbol{s}}(\tau)$, the periodic correlation function $\overset{\text{P}}{\theta}_{\boldsymbol{r},\boldsymbol{s}}(\tau)$, and the odd correlation function $\overset{\text{O}}{\theta}_{\boldsymbol{r},\boldsymbol{s}}(\tau)$ have the same value. This implies the following:

For $|\tau| < \ell_0 + 1$,

$$\overset{\text{A}}{\theta}_{\boldsymbol{r},\boldsymbol{s}}(\tau) = \overset{\text{P}}{\theta}_{\boldsymbol{r},\boldsymbol{s}}(\tau) = \overset{\text{O}}{\theta}_{\boldsymbol{r},\boldsymbol{s}}(\tau). \tag{3}$$

Here a k-element shift $\boldsymbol{S}(k;\ \boldsymbol{v}_r)$ of a sequence $\boldsymbol{v}_r$ of length ℓ_v is defined as follows:

$$\boldsymbol{S}(k;\ \boldsymbol{v}_r) = [v_{r,k}, v_{r,k+1}, \ldots, v_{r,\ell_v-1}, v_{r,0}, \ldots, v_{r,k-1}]. \tag{4}$$

Sequence shifting is used for the generation of the proposed sequence set.

The correlation functions of $\boldsymbol{S}(j;\ \boldsymbol{v}_r)$ and $\boldsymbol{S}(k;\ \boldsymbol{v}_s)$ satisfy the following [12]:

$$\overset{\text{P}}{\theta}_{\boldsymbol{S}(j;\,\boldsymbol{v}_r),\boldsymbol{S}(k;\,\boldsymbol{v}_s)}(\tau) = \overset{\text{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(k - j + \tau). \tag{5}$$

2.1 Sequence Interleaving

An interleaved pair of sequences $\boldsymbol{I}^{(+)}(\boldsymbol{v}_r, \boldsymbol{v}_s)$ and $\boldsymbol{I}^{(-)}(\boldsymbol{v}_r, \boldsymbol{v}_s)$ of a pair of sequences $\boldsymbol{v}_r$ and $\boldsymbol{v}_s$, each of length ℓ_v, is defined as follows:

$$\boldsymbol{I}^{(+)}(\boldsymbol{v}_r, \boldsymbol{v}_s) = [\underbrace{v_{r,0},\ v_{s,0},\ v_{r,1},\ v_{s,1},\ \cdots,\ v_{r,\ell_v-1},\ v_{s,\ell_v-1}}_{2\ell_v}], \tag{6a}$$

$$\boldsymbol{I}^{(-)}(\boldsymbol{v}_r, \boldsymbol{v}_s) = [\underbrace{v_{r,0},\ -v_{s,0}, v_{r,1},\ -v_{s,1},\ \cdots,\ v_{r,\ell_v-1},\ -v_{s,\ell_v-1}}_{2\ell_v}]. \tag{6b}$$

The correlation functions of $\boldsymbol{I}^{(+)}(\boldsymbol{v}_r, \boldsymbol{v}_s)$ and $\boldsymbol{I}^{(-)}(\boldsymbol{v}_{r'}, \boldsymbol{v}_{s'})$ satisfy the following [12]:

$$\overset{\text{P}}{\theta}_{\boldsymbol{I}^{(+)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(+)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau) = \overset{\text{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{r'}}(\tau) + \overset{\text{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{s'}}(\tau), \tag{7a}$$

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(+)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(+)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau+1)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{s'}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{r'}}(\tau+1), \qquad (7b)\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(-)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(-)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{r'}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{s'}}(\tau), \qquad (7c)\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(-)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(-)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau+1)\\
&= -\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{s'}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{r'}}(\tau+1), \qquad (7d)\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(+)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(-)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{r'}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{s'}}(\tau), \qquad (7e)\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(+)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(-)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau+1)\\
&= -\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{s'}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{r'}}(\tau+1), \qquad (7f)\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(-)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(+)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{r'}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{s'}}(\tau), \qquad (7g)\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{I}^{(-)}(\boldsymbol{v}_r,\boldsymbol{v}_s),\boldsymbol{I}^{(+)}(\boldsymbol{v}_{r'},\boldsymbol{v}_{s'})}(2\tau+1)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_{s'}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\boldsymbol{v}_{r'}}(\tau+1). \qquad (7h)
\end{aligned}
$$

Equations (7a)–(7h) are used to ensure that the proposed sequences have zcz.

2.2 Sequence Concatenation

We use a sequence concatenation $\boldsymbol{C}(\boldsymbol{a}_j, \boldsymbol{v}_r)$ of sequence $\boldsymbol{v}_r$ of length ℓ_v with a sequence $\boldsymbol{a}_j$ of length ℓ_a as follows:

$$
\begin{aligned}
&\boldsymbol{C}(\boldsymbol{a}_j, \boldsymbol{v}_r)\\
&= [C(\boldsymbol{a}_j, \boldsymbol{v}_r)_0, C(\boldsymbol{a}_j, \boldsymbol{v}_r)_1, \ldots C(\boldsymbol{a}_j, \boldsymbol{v}_r)_{\ell_a\ell_v-1}]\\
&= [a_{j,0}\boldsymbol{v}_r, \ldots, a_{j,\ell_a-1}\boldsymbol{v}_r]\\
&= \left.\begin{array}{l}
[\,\underbrace{a_{j,0}v_{r,0}, \ldots, a_{j,0}v_{r,\ell_v-1},}_{\ell_v}\\
\underbrace{a_{j,1}v_{r,0}, \ldots, a_{j,1}v_{r,\ell_v-1},}_{\ell_v}\ \ldots,\\
\underbrace{a_{j,\ell_a-1}v_{r,0}, \ldots, a_{j,\ell_a-1}v_{r,\ell_v-1}}_{\ell_v}\,]
\end{array}\right\}\ell_a. \qquad (8a)
\end{aligned}
$$

From the above definition of $\boldsymbol{C}(\boldsymbol{a}_j, \boldsymbol{v}_r)$, we have the following:

For $i = \ell_v i' + i''$,

$$C(a_{j,}, \boldsymbol{v}_r)_i = C(a_{j,}, \boldsymbol{v}_r)_{\ell_v i' + i''} = a_{j,i'}v_{r,i''}. \qquad (8b)$$

The correlation functions of $\boldsymbol{C}(\boldsymbol{a}_j, \boldsymbol{v}_r)$ and $\boldsymbol{C}(\boldsymbol{a}_k, \boldsymbol{v}_s)$ satisfy the following:

For $\tau = \ell_v\tau' + \tau''$,

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{C}(\boldsymbol{a}_j,\boldsymbol{v}_r),\boldsymbol{C}(\boldsymbol{a}_k,\boldsymbol{v}_s)}(\tau)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{C}(\boldsymbol{a}_j,\boldsymbol{v}_r),\boldsymbol{C}(\boldsymbol{a}_k,\boldsymbol{v}_s)}(\ell_v\tau' + \tau'')\\
&= \sum_{i=0}^{\ell_a\ell_v-1} C(\boldsymbol{a}_j, \boldsymbol{v}_r)_i C(\boldsymbol{a}_k, \boldsymbol{v}_s)_{i+\ell_v\tau'+\tau''}\\
&= \sum_{i'=0}^{\ell_a-1}\sum_{i''=0}^{\ell_v-1}\Big(C(\boldsymbol{a}_j, \boldsymbol{v}_r)_{(\ell_v i'+i'')}\\
&\qquad C(\boldsymbol{a}_k, \boldsymbol{v}_s)_{(\ell_v(i'+\tau')+(i''+\tau''))}\Big)\\
&= \sum_{i'=0}^{\ell_a-1}\sum_{i''=0}^{\ell_v-1}\Big(a_{j,i'}v_{r,i''}\\
&\qquad a_{j,(((\ell_v(i'+\tau')+(i''+\tau''))\oslash\ell_v)\bmod \ell_a)}\\
&\qquad v_{s,((\ell_v(i'+\tau')+(i''+\tau''))\bmod \ell_v)}\Big). \qquad (9a)
\end{aligned}
$$

We also have the following:

$$
\begin{aligned}
&(\ell_v(i'+\tau') + (i''+\tau''))\oslash\ell_v\\
&= \begin{cases} i'+\tau'' & \text{if } 0 \le i''+\tau'' < \ell_v\\ i'+\tau''+1 & \text{if } \ell_v \le i''+\tau'' < 2\ell_v. \end{cases} \qquad (9b)
\end{aligned}
$$

Putting these together, we obtain the following:

For $\tau = \ell_v\tau' + \tau''$,

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{C(a_j,v_r),C(a_k,v_s)}(\tau)\\
&= \sum_{i'=0}^{\ell_a-\tau'-1}\left(\sum_{i''=0}^{\ell_v-\tau''-1} a_{j,i'}a_{k,(i'+\tau')}v_{r,i''}v_{s,(i''+\tau'')}\right)\\
&+ \sum_{i'=\ell_a-\tau'}^{\ell_a-1}\left(\sum_{i''=0}^{\ell_v-\tau''-1} a_{j,i'}a_{k,(i'+\tau'-\ell_a)}v_{r,i''}v_{s,(i''+\tau'')}\right)\\
&+ \sum_{i'=0}^{\ell_a-\tau'-1}\left(\sum_{i''=\ell_v-\tau''}^{\ell_v-1}\Big(a_{j,i'}a_{k,(i'+\tau'+1)}\right.\\
&\qquad \left. v_{r,i''}v_{s,(i''+\tau''-\ell_v)}\Big)\right)\\
&+ \sum_{i'=\ell_a-\tau'}^{\ell_a-1}\left(\sum_{i''=\ell_v-\tau''}^{\ell_v-1}\Big(a_{j,i'}a_{k,(i'+\tau'+1-\ell_a)}\right.\\
&\qquad \left. v_{r,i''}v_{s,(i''+\tau''-\ell_v)}\Big)\right)\\
&= \sum_{i'=0}^{\ell_a-1} a_{j,i'}a_{k,(i'+\tau')}\sum_{i''=0}^{\ell_v-\tau''-1} v_{r,i''}v_{s,(i''+\tau'')}\\
&+ \sum_{i'=0}^{\ell_a-1} a_{j,i'}a_{k,((i'+\tau'+1)\bmod \ell_a)}\sum_{i''=\ell_v-\tau''}^{\ell_v-1} v_{r,i''}v_{s,(i''+\tau''-\ell_v)}\\
&= \sum_{i'=0}^{\ell_a-1} a_{j,i'}a_{k,(i'+\tau')}\overset{\mathrm{A}}{\theta}_{v_r,v_s}(\tau'')
\end{aligned}
$$

$$+\sum_{i'=0}^{\ell_a-1} a_{j,i'} a_{k,((i'+\tau'+1) \bmod \ell_a)} \overset{\mathrm{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau''-\ell_v). \tag{9c}$$

Consequently, we have the following:

$$\begin{aligned}
&\text{For } \tau = \ell_v\tau' + \tau'',\\
&\overset{\mathrm{P}}{\theta}_{C(\boldsymbol{a}_j,\boldsymbol{v}_r),C(\boldsymbol{a}_k,\boldsymbol{v}_s)}(\tau)\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{a}_j,\boldsymbol{a}_k}(\tau')\overset{\mathrm{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau'')\\
&+ \overset{\mathrm{P}}{\theta}_{\boldsymbol{a}_j,\boldsymbol{a}_k}(\tau'+1)\overset{\mathrm{A}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau''-\ell_v).
\end{aligned} \tag{9d}$$

Equation (9d) is used to ensure that the proposed sequence set has a wide intra-subset zcz.

2.3 zcz Sequence Set

If a length-ℓ_v sequence set $\{\boldsymbol{v}_r | 0 \le r < N\}$ satisfies the following conditions, then the sequence set has a zcz and is denoted by $Z(\ell_v, N, z)$ [6].

- All of the periodic auto-correlation functions of $\boldsymbol{v}_r$, $\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_r}(\tau)$, are zero when the absolute value of the phase shift $|\tau|$ is not zero and $|\tau|$ is less than or equal to a specified integer z.
- All of the periodic cross-correlation functions of $\boldsymbol{v}_r$ and $\boldsymbol{v}_s$, $\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau)$, are zero when the absolute value of the phase shift $|\tau|$ is less than or equal to a specified integer z.

These conditions are formulated as follows:

$$\begin{aligned}
&\forall r, \forall \tau, 0 < |\tau| \le z,\\
&\quad \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_r}(\tau) = 0, \quad \text{and} && (10a)\\
&\forall r \ne s, \forall \tau, |\tau| \le z,\\
&\quad \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\boldsymbol{v}_s}(\tau) = 0. && (10b)
\end{aligned}$$

We denote the zcz sequence set for aperiodic correlation functions by $\overset{\mathrm{A}}{Z}(\ell_v, N, z)$ and the zcz sequence set for odd correlation functions by $\overset{\mathrm{O}}{Z}(\ell_v, N, z)$. From the definition of the zcz sequence set, the rows of the Hadamard matrix of order ℓ_h are $Z(n, n, 0)$.

A zcz sequence set satisfying the theoretical bound on the sequence member size and the sequence period is called an optimal zcz sequence set [5], [14], [22]–[24], [27]–[29], [31], [36], [41], [42], [49]–[51].

2.4 Perfect Sequence

A perfect sequence $\boldsymbol{p} = [p_0, p_1, \ldots, p_{\ell_p-1}]$ of period ℓ_p is a sequence that satisfies the following:

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{p},\boldsymbol{p}}(\tau) = \begin{cases} |\boldsymbol{p}| & \text{if } \tau \equiv 0 \pmod{\ell_p},\\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

3. Sequence Construction

A novel scheme for the construction of a set of ternary sequences with a zcz is presented in this section.

For a given Hadamard matrix $H = (h_{i,j})$ of order ℓ_h and a binary or ternary perfect sequence of length ℓ_p, the proposed scheme can recursively construct a series of sets $\boldsymbol{U}^{(m)} = \{\boldsymbol{u}_r^{(m,j)} | 0 \le j < \ell_p, 0 \le r < 2\ell_h\}$ for $m \ge 0$. The proposed sequence set is $\boldsymbol{U}^{(m)}$ for $m \ge 0$, which consists of ℓ_p subsets of the sequence, each of length $2^{m+2}(\ell_h+1)\ell_p$ as follows:

$$\begin{aligned}
&\text{For } m \ge 0,\\
&\quad \boldsymbol{U}^{(m)} = \bigcup_{j=0}^{\ell_p-1} \boldsymbol{U}^{(m,j)}. && (12a)\\
&\text{For } m \ge 0, \quad 0 \le j \ne k < \ell_p,\\
&\quad \boldsymbol{U}^{(m,j)} \cap \boldsymbol{U}^{(m,k)} = \emptyset. && (12b)\\
&\text{For } m \ge 0, \quad 0 \le j < \ell_p,\\
&\quad \boldsymbol{U}^{(m,j)} = \left\{\boldsymbol{u}_r^{(m,j)} | 0 \le r < 2\ell_h\right\}. && (12c)
\end{aligned}$$

The proposed sequences $\boldsymbol{u}_r^{(m,k)}$ for $0 \le k < \ell_p$ and $0 \le r < 2\ell_h$ are constructed as follows.

First, sequences $\boldsymbol{w}_r$ of length $\ell_w = 2(\ell_h+1)$ are generated for $0 \le r < 2\ell_h$ by the concatenation of the r-th row of the Hadamard matrix H denoted by $\boldsymbol{h}_r = [h_{r,s} | 0 \le s < \ell_h]$. The inner product of the rows $\boldsymbol{h}_r$ and $\boldsymbol{h}_s$ satisfies the following:

$$\begin{aligned}
&\text{For } r \ne s,\\
&\quad \boldsymbol{h}_r \cdot \boldsymbol{h}_s = 0.
\end{aligned} \tag{13}$$

The sequences $\boldsymbol{w}_r$ are generated by the concatenation of $\boldsymbol{h}_r$ and zero filling as follows:

$$\begin{aligned}
&\text{For } 0 \le r < \ell_h,\\
&\quad \boldsymbol{w}_{2r+0} = [\boldsymbol{h}_r, 0, \boldsymbol{h}_r, 0]\\
&\quad = [h_{r,0}, \ldots, h_{r,\ell_h-1}, 0, h_{r,0}, \ldots, h_{r,\ell_h-1}, 0], && (14a)\\
&\quad \boldsymbol{w}_{2r+1} = [\boldsymbol{h}_r, 0, -\boldsymbol{h}_r, 0]\\
&\quad = [h_{r,0}, \ldots, h_{r,\ell_h-1}, 0, -h_{r,0}, \ldots, -h_{r,\ell_h-1}, 0]. && (14b)
\end{aligned}$$

The length of the sequence $\boldsymbol{w}_r$ is denoted by ℓ_w and is equal to $2(\ell_h+1)$. The generated $\boldsymbol{w}_{2r+0}$ and $\boldsymbol{w}_{2s+1}$ satisfy the following:

$$\begin{aligned}
&\text{For } 0 \le r, \ s < \ell_h, \ \forall\tau,\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+0},\boldsymbol{w}_{2s+1}}(\tau)\\
&= (\overset{\mathrm{A}}{\theta}_{[\boldsymbol{h}_r,0],[\boldsymbol{h}_s,0]}(\tau) + \overset{\mathrm{A}}{\theta}_{[\boldsymbol{h}_r,0],[\boldsymbol{h}_s,0]}(\tau-(\ell_h+1)))\\
&- (\overset{\mathrm{A}}{\theta}_{[\boldsymbol{h}_r,0],[\boldsymbol{h}_s,0]}(\tau) - \overset{\mathrm{A}}{\theta}_{[\boldsymbol{h}_r,0],[\boldsymbol{h}_s,0]}(\tau-(\ell_h+1))) = 0.
\end{aligned} \tag{15a}$$

The correlation functions of $\boldsymbol{w}_{2r+0}$ and $\boldsymbol{w}_{2s+0}$, and those of $\boldsymbol{w}_{2r+1}$ and $\boldsymbol{w}_{2s+1}$, satisfy the following:

For $0 \le r,\ s < \ell_h$,

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+0},\boldsymbol{w}_{2s+0}}(0) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+1},\boldsymbol{w}_{2s+1}}(0)\\
&= 2\boldsymbol{h}_r \cdot \boldsymbol{h}_s\\
&= \begin{cases} 0 & \text{if } r = s,\\ 2\ell_h & \text{otherwise.}\end{cases}
\end{aligned}
\tag{15b}
$$

For $0 \le r,\ s < \ell_h$,

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+0},\boldsymbol{w}_{2s+0}}(1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+1},\boldsymbol{w}_{2s+1}}(1) =\\
&= (\overset{\mathrm{A}}{\theta}_{[\boldsymbol{h}_r,0],[\boldsymbol{h}_s,0]}(1) + \overset{\mathrm{A}}{\theta}_{[\boldsymbol{h}_r,0],[\boldsymbol{h}_s,0]}(1))\\
&= 2\left(\sum_{i=0}^{\ell_h-2} h_{r,i}h_{s,i+1} + h_{r,\ell_h-1}\cdot 0\right)\\
&= 2\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\boldsymbol{h}_s}(1),
\end{aligned}
\tag{15c}
$$

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+0},\boldsymbol{w}_{2s+0}}(-1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{w}_{2r+1},\boldsymbol{w}_{2s+1}}(-1) =\\
&= 2\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\boldsymbol{h}_s}(-1).
\end{aligned}
\tag{15d}
$$

Next, sequences $\boldsymbol{z}_r^{(j)}$ are generated for $0 \le k < \ell_p$ and $0 \le r < 2\ell_h$ by the concatenation of $\boldsymbol{w}_r$ with j-shifted elements of $\boldsymbol{p}$, ($\boldsymbol{S}(j;\ \boldsymbol{p}) = [p_j, p_{j+1}, \ldots, p_{\ell_p-1}, p_0, \ldots, p_{j-1}]$) as the coefficients to $\boldsymbol{w}_r$ as follows:

For $0 \le j < \ell_p,\ \ 0 \le r < 2\ell_h$,

$$
\begin{aligned}
\boldsymbol{z}_r^{(j)} &= \boldsymbol{C}(\boldsymbol{S}(j;\ \boldsymbol{p}), \boldsymbol{w}_r)\\
&= [\underbrace{p_j\boldsymbol{w}_r,\ p_{j+1}\boldsymbol{w}_r,\ \cdots,\ p_{\ell_p-1}\boldsymbol{w}_r,\ p_0\boldsymbol{w}_r,\ \cdots,\ p_{j-1}\boldsymbol{w}_r}_{\ell_w\ell_p = 2(\ell_h+1)\ell_p}].
\end{aligned}
\tag{16}
$$

The length of $\boldsymbol{z}_r^{(j)}$ is denoted by ℓ_z and is equal to $2(\ell_h+1)\ell_p$. From Eqs. (9d), (15), and (16), the correlation functions of the constructed sequences $\boldsymbol{z}_r^{(j)}$ and $\boldsymbol{z}_r^{(k)}$ satisfy the following:

For $0 \le j,\ k < \ell_p,\ \ 0 \le r,\ s < 2\ell_h,\ \ \tau = \ell_w\tau' + \tau''$,

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(\ell_w\tau' + \tau'')\\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{p},\boldsymbol{p}}(j-(k+\tau'))\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau'')\\
&+ \overset{\mathrm{P}}{\theta}_{\boldsymbol{p},\boldsymbol{p}}(j-(k+\tau'+1))\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau''-\ell_w),\\
&= \begin{cases}
|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau - \ell_w(j-k)) \\
\qquad \text{if } \tau' = j-k\,,\\
|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau - \ell_w(j-k-1)-\ell_w) \\
\qquad \text{if } \tau' = j-k-1,\\
0 \qquad \text{otherwise,}
\end{cases}\\
&= \begin{cases}
|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau - \ell_w(j-k)), \\
\qquad \text{if } \ell_w|j-k| \le \tau < \ell_w(|j-k|+1)\\
|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau - \ell_w(j-k)) \\
\qquad \text{if } \ell_w(|j-k|-1) \le \tau < \ell_w|j-k|,\\
0 \qquad \text{otherwise,}
\end{cases}\\
&= \begin{cases}
|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau - \ell_w(j-k)) \\
\qquad \text{if } |j-k| \le \tau \oslash \ell_w < |j-k|+1\,,\\
|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{w}_r,\boldsymbol{w}_s}(\tau - \ell_w(j-k)) \\
\qquad \text{if } |j-k|-1 \le \tau \oslash \ell_w < |j-k|,\\
0 \qquad \text{otherwise.}
\end{cases}
\end{aligned}
\tag{17}
$$

From Eq. (17), we have the following:

For $1 < |j-k| < \ell_p - 1,\ \ |\tau| \le \ell_w(|j-k|-1)$,

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(\tau) = 0.
\tag{18}
$$

From Eqs. (7) and (15), we obtain the following:

For $|\tau| \le 1$,

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r+0}^{(j)},\boldsymbol{z}_{2s+0}^{(k)}}(\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r+1}^{(j)},\boldsymbol{z}_{2s+1}^{(k)}}(\tau),
\tag{19a}
$$

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r+0}^{(j)},\boldsymbol{z}_{2s+0}^{(k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r+1}^{(j)},\boldsymbol{z}_{2s+1}^{(k)}}(2) = 2\,|\boldsymbol{p}|\,\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\boldsymbol{h}_s}(1),
\tag{19b}
$$

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r+0}^{(j)},\boldsymbol{z}_{2s+1}^{(k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r+1}^{(j)},\boldsymbol{z}_{2s+0}^{(k)}}(2) = 0.
\tag{19c}
$$

Then, sequences $\boldsymbol{z}_r^{(k)}$ also satisfy the following:

For $j = k$, and $r = s,\ 0 < |\tau| \le 1$,

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(\tau) = 0.
\tag{20a}
$$

For $j \ne k$, or $r \ne s,\ 0 \le |\tau| \le 1$,

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(\tau) = 0.
\tag{20b}
$$

For $1 < |j-k| < \ell_p - 1, |\tau| \le \ell_w(|j-k|-1)$,

$$
\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(\tau) = 0.
\tag{20c}
$$

For a pair of even numbers $r = 2\rho$ and $s = 2\sigma$, we also have the following:

For $0 \le j, k < \ell_p,\ r = 2\rho + 0, s = 2\sigma$,

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_s^{(k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2\rho+0}^{(j)},\boldsymbol{z}_{2\sigma+0}^{(k)}}(2)\\
&\quad= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{r+1}^{(j)},\boldsymbol{z}_{s+1}^{(k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2\rho+1}^{(j)},\boldsymbol{z}_{2\sigma+1}^{(k)}}(2),
\end{aligned}
\tag{20d}
$$

$$
\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_r^{(j)},\boldsymbol{z}_{s+1}^{(k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2\rho+0}^{(j)},\boldsymbol{z}_{2\sigma+1}^{(k)}}(2)\\
&\quad= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{r+1}^{(j)},\boldsymbol{z}_{s}^{(k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2\rho+1}^{(j)},\boldsymbol{z}_{2\sigma+0}^{(k)}}(2) = 0.
\end{aligned}
\tag{20e}
$$

Next, a sequence pair $\boldsymbol{u}_{2r+0}^{(0,k)}$ and $\boldsymbol{u}_{2r+1}^{(0,k)}$ is generated from the sequence pair $\boldsymbol{z}_{2r+0}^{(k)}$ and $\boldsymbol{z}_{2r+1}^{(k)}$ by using sequence interleaving. Each sequence has a run of zero elements at its tail. For simplicity, we denote the length, member size, and length of the run of zero elements at the tail of $\boldsymbol{z}_r^{(k)}$ by $L^{(m)}$, $N^{(m)}$, and $T^{(m)}$, respectively. The sequence pair $\boldsymbol{u}_{2r+0}^{(0,k)}$ and $\boldsymbol{u}_{2r+1}^{(0,k)}$ is generated from $\boldsymbol{z}_{2r+0}^{(k)}$ and $\boldsymbol{z}_{2r+1}^{(k)}$ as follows:

For $0 \le r < \ell_h$,

$$\boldsymbol{u}_{2r+0}^{(0,k)} = \boldsymbol{I}^{(+)}(\boldsymbol{z}_{2r+0}^{(k)}, \boldsymbol{z}_{2r+1}^{(k)})$$
$$= [\underbrace{z_{2r+0,0}^{(k)},\ z_{2r+1,0}^{(k)},\ \cdots,\ z_{2r+0,L^{(m)}-1}^{(k)},\ z_{2r+1,L^{(m)}-1}^{(k)}}_{2\ell_z=2\ell_w\ell_p}], \tag{21a}$$

$$\boldsymbol{u}_{2r+1}^{(0,k)} = \boldsymbol{I}^{(-)}(\boldsymbol{z}_{2r+0}^{(k)}, \boldsymbol{z}_{2r+1}^{(k)})$$
$$= [\underbrace{z_{2r+0,0}^{(k)},\ -z_{2r+1,0}^{(k)},\ \cdots,\ z_{2r+0,L^{(m)}-1}^{(k)},\ -z_{2r+1,L^{(m)}-1}^{(k)}}_{2\ell_z=2\ell_w\ell_p}]. \tag{21b}$$

From Eqs. (7), (21a), and (21b), we have as follows:

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 0), s(= 2s' + 0) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+0}^{(0,k)}}(2\tau)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau), \tag{22a}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+0}^{(0,k)}}(2\tau+1)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau+1). \tag{22b}$$

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 1), s(= 2s' + 1) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+1}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2\tau)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau), \tag{22c}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+1}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2\tau+1)$$
$$= -\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau+1). \tag{22d}$$

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 0), s(= 2s' + 1) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2\tau)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau), \tag{22e}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2\tau+1)$$
$$= -\overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau+1). \tag{22f}$$

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 1), s(= 2s' + 0) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+1}^{(0,j)},\boldsymbol{u}_{2s'+0}^{(0,k)}}(2\tau)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau), \tag{22g}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+1}^{(0,j)},\boldsymbol{u}_{2s'+0}^{(0,k)}}(2\tau+1)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+0}^{(j)},\boldsymbol{z}_{2s'+1}^{(k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{z}_{2r'+1}^{(j)},\boldsymbol{z}_{2s'+0}^{(k)}}(\tau+1). \tag{22h}$$

Then, the correlation functions of the sequences $\boldsymbol{u}_r^{(0,k)}$ also satisfy the following:

For $(j,\ r) = (k,\ s), 0 < |\tau| \le 1, 0 \le r, s < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(\tau) = 0. \tag{23a}$$

For $(j,\ r) \ne (k,\ s), 0 \le |\tau| \le 1, 0 \le r, s < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(\tau) = 0. \tag{23b}$$

For $1 < |j-k| < \ell_p - 1, |\tau| \le 2\ell_w(|j-k| - 1), 0 \le r,$
$s < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(\tau) = 0. \tag{23c}$$

The correlation functions of the sequences $\boldsymbol{u}_r^{(0,k)}$ also satisfy the following:

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 0), s(= 2s' + 0) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+0}^{(0,k)}}(2)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+1}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2). \tag{24a}$$

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 0), s(= 2s' + 1) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+0}^{(0,j)},\boldsymbol{u}_{2s'+1}^{(0,k)}}(2) = 0. \tag{24b}$$

For $0 \le j, k < \ell_p,\ 0 \le r(= 2r' + 1), s(= 2s' + 0) < 2\ell_h$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(0,j)},\boldsymbol{u}_s^{(0,k)}}(2)$$
$$= \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_{2r'+1}^{(0,j)},\boldsymbol{u}_{2s'+0}^{(0,k)}}(2) = 0. \tag{24c}$$

Next, a sequence pair $\boldsymbol{u}_{2r+0}^{(m+1,k)}$ and $\boldsymbol{u}_{2r+1}^{(m+1,k)}$ is generated from the sequence pair $\boldsymbol{u}_{2r+0}^{(m,k)}$ and $\boldsymbol{u}_{2r+1}^{(m,k)}$ by using sequence interleaving. Each sequence has a run of zero elements at its tail. For simplicity, we denote the length, member size, and length of the run of zero elements at the tail of $\boldsymbol{u}_r^{(m,k)}$ by $L^{(m)}$, $N^{(m)}$, and $T^{(m)}$, respectively. The length $L^{(0)}$, member size $N^{(0)}$, and length $T^{(0)}$ of the tail run of zero elements of the proposed sequence $\boldsymbol{u}_r^{(m,k)}$ satisfy

$$L^{(0)} = 2\ell_w\ell_p = 4(\ell_h + 1)\ell_p, \tag{25a}$$
$$N^{(0)} = 2\ell_h\ell_p, \tag{25b}$$
$$T^{(0)} = 2. \tag{25c}$$

The sequence pair $\boldsymbol{u}_{2r+0}^{(m+1,k)}$ and $\boldsymbol{u}_{2r+1}^{(m+1,k)}$ is generated from $\boldsymbol{u}_{2r+0}^{(m,k)}$ and $\boldsymbol{u}_{2r+1}^{(m,k)}$ as follows:

For $0 \le r < \ell_h$,

$$\boldsymbol{u}_{2r+0}^{(m+1,k)} = \boldsymbol{I}^{(+)}(\boldsymbol{u}_{2r+0}^{(m,k)}, \boldsymbol{u}_{2r+1}^{(m,k)})$$
$$= [\underbrace{u_{2r+0,0}^{(m,k)},\ u_{2r+1,0}^{(m,k)},\ \cdots,\ u_{2r+0,L^{(m)}-1}^{(m,k)},\ u_{2r+1,L^{(m)}-1}^{(m,k)}}_{2L^{(m)}}], \tag{26a}$$

$$\boldsymbol{u}_{2r+1}^{(m+1,k)} = \boldsymbol{I}^{(-)}(\boldsymbol{u}_{2r+0}^{(m,k)}, \boldsymbol{u}_{2r+1}^{(m,k)})$$
$$= [\underbrace{u_{2r+0,0}^{(m,k)},\ -u_{2r+1,0}^{(m,k)},\ \cdots,\ u_{2r+0,L^{(m)}-1}^{(m,k)},\ -u_{2r+1,L^{(m)}-1}^{(m,k)}}_{2L^{(m)}}]. \tag{26b}$$

From Eqs. (7), (26a), and (26b), we have as follows:

For $0 \leq j, k < \ell_p$, $0 \leq r, s < \ell_h$,

$$\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m+1,j)}, u_{2s+0}^{(m+1,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau), \tag{27a}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m+1,j)}, u_{2s+0}^{(m+1,k)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau+1), \tag{27b}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m+1,j)}, u_{2s+1}^{(m+1,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau), \tag{27c}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m+1,j)}, u_{2s+1}^{(m+1,k)}}(2\tau+1) = -\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau+1), \tag{27d}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m+1,j)}, u_{2s+1}^{(m+1,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau), \tag{27e}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m+1,j)}, u_{2s+1}^{(m+1,k)}}(2\tau+1) = -\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau) + \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau+1), \tag{27f}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m+1,j)}, u_{2s+0}^{(m+1,k)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau), \tag{27g}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m+1,j)}, u_{2s+0}^{(m+1,k)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+1}^{(m,k)}}(\tau) - \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+0}^{(m,k)}}(\tau+1). \tag{27h}$$

Then, sequences $\boldsymbol{u}_r^{(m,k)}$ also satisfy the following:

For $j = k$, and, $r = s, 0 < |\tau| \leq 2^{m+1} - 1$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,j)}, u_s^{(m,k)}}(\tau) = 0, \tag{28a}$$

For $j \neq k$, or $r \neq s, 0 \leq |\tau| \leq 2^{m+1} - 1$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,j)}, u_s^{(m,k)}}(\tau) = 0, \tag{28b}$$

For $1 < |j-k| < \ell_p - 1, |\tau| \leq 2^{m+1}\ell_w(|j-k|-1)$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,j)}, u_s^{(m,k)}}(\tau) = 0. \tag{28c}$$

For $0 \leq j, k < \ell_p$, $0 \leq r, s < \ell_h$,

$$\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+0}^{(m,k)}}(2) = \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+1}^{(m,k)}}(2). \tag{28d}$$

$$\overset{\mathrm{P}}{\theta}_{u_{2r+0}^{(m,j)}, u_{2s+1}^{(m,k)}}(2) = \overset{\mathrm{P}}{\theta}_{u_{2r+1}^{(m,j)}, u_{2s+0}^{(m,k)}}(2) = 0. \tag{28e}$$

From Eqs. (26a), (26b), and (27), the length $L^{(m)}$, member size $N^{(m)}$, and length $T^{(m)}$ of the tail run of zero elements of the proposed sequence $\boldsymbol{u}_r^{(m,k)}$ satisfy

$$L^{(m+1)} = 2L^{(m)}, \tag{29a}$$

$$N^{(m+1)} = N^{(m)}, \text{ and} \tag{29b}$$

$$T^{(m+1)} = 2T^{(m)}. \tag{29c}$$

From Eqs. (25) and (29), we obtain

$$L^{(m)} = 2^{m+2}(\ell_h + 1)\ell_p, \tag{30a}$$

$$N^{(m)} = 2\ell_h\ell_p, \text{ and} \tag{30b}$$

$$T^{(m)} = 2^{m+1}. \tag{30c}$$

Therefore, the proposed sequence $\boldsymbol{u}_r^{(m,k)}$ has $T^{(m)} = 2^{m+1}$ zero-value elements in its tail, as follows:

For $L^{(m)} - T^{(m)} - 1 \leq j < L^{(m)}$,

$$u_{r,j}^{(m,k)} = 0. \tag{31}$$

4. Properties of the Proposed Sequence Set

From Eq. (3), when the absolute value of phase shift $|\tau|$ is less than or equal to $T^{(m)} + 1 (= 2^{m+1} + 1)$, the aperiodic correlation function $\overset{\mathrm{A}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau)$, the periodic correlation function $\overset{\mathrm{P}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau)$, and the odd correlation function $\overset{\mathrm{O}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau)$ have the same value. This implies the following theorem:

Theorem 1:

For $0 \leq k < \ell_p$,

$0 \leq r,\ s, < 2\ell_h,\ \forall|\tau| \leq T^{(m)} + 1 = 2^{m+1} + 1$,

$$\overset{\mathrm{A}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau) = \overset{\mathrm{P}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau) = \overset{\mathrm{O}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau). \tag{32}$$

Sequence set $\{\boldsymbol{u}_r^{(m,k)}\}$ has a zcz for the periodic correlation function $\overset{\mathrm{P}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau)$, the aperiodic correlation function $\overset{\mathrm{A}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau)$, and the odd correlation function $\overset{\mathrm{O}}{\theta}_{u_r^{(m,k)}, u_s^{(m,k)}}(\tau)$ for phase shift τ, which implies a second theorem, as follows [14]:

Theorem 2: The periodic correlation function and the aperiodic correlation function of $\{\boldsymbol{u}_r^{(m,k)}\}$ have a zcz from $-(2^{m+1} - 1)$ to $2^{m+1} - 1$. That is, the following holds:

For $0 \leq k < \ell_p$,

$0 \leq r < 2\ell_h,\ \forall|\tau| \leq 2^{m+1} + 1$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,k)}, u_r^{(m,k)}}(\tau) = \overset{\mathrm{A}}{\theta}_{u_r^{(m,k)}, u_r^{(m,k)}}(\tau) = \overset{\mathrm{O}}{\theta}_{u_r^{(m,k)}, u_r^{(m,k)}}(\tau) = 0. \tag{33a}$$

For $0 \leq k < \ell_p$,

$0 \leq r \neq s, < 2\ell_h,\ \forall|\tau| \leq 2^{m+1} + 1$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,k)},u_s^{(m,k)}}(\tau) = \overset{\mathrm{A}}{\theta}_{u_r^{(m,k)},u_s^{(m,k)}}(\tau)$$
$$= \overset{\mathrm{O}}{\theta}_{u_r^{(m,k)},u_s^{(m,k)}}(\tau) = 0. \quad (33b)$$

For $0 \le k \ne k' < \ell_p$,

$$0 \le r,\ s, < 2\ell_h,\ \forall|\tau| \le 2^{m+1} + 1,$$
$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,k)},u_s^{(m,k')}}(\tau) = \overset{\mathrm{A}}{\theta}_{u_r^{(m,k)},u_s^{(m,k')}}(\tau)$$
$$= \overset{\mathrm{O}}{\theta}_{u_r^{(m,k)},u_s^{(m,k')}}(\tau) = 0. \quad (33c)$$

Theorem 2 indicates that the sequence set $\{\boldsymbol{u}_r^{(m,k)}\}$ is $Z(2^{m+2}(\ell_h+1)\ell_p,\ 2\ell_h\ell_p,\ 2^{m+1}-1)$. For example, the proposed sequence construction can produce a set of sequences, each of length 240 ($= 2^{1+2}(4+1)\times 6$), from an Hadamard matrix of order $\ell_h = 4$ and a binary/ternary perfect sequence of length $\ell_p = 6$ for $m = 1$.

Since the proposed sequence set is constructed from an Hadamard matrix without any restrictions, various types of sequence sets can be constructed according to the construction of the Hadamard matrices [54]. The proposed sequence construction is an extension of the sequence construction reported in [16], [19].

4.1 Performance of the Proposed Sequence Set Construction

Since the theoretical upper bound on the sequence member size of a $Z(L, N, Z)$ sequence is $\frac{L}{Z+1}$ [32], [41], [45], [46], the ratio $\frac{N(Z+1)}{L}$ indicates the performance of the $Z(L, N, Z)$ sequence set. We denote this parameter by ϵ. Also, since the proposed sequence set is $Z(2^{m+2}(\ell_h+1)\ell_p,\ 2\ell_h\ell_p,\ 2^{m+1}-1)$, parameter ϵ of the sequence set is equal to $\frac{2^{m+2}\ell_h\ell_p}{2^{m+2}(\ell_h+1)\ell_p} = \frac{\ell_h}{\ell_h+1}$.

An effective index of the performance of a ternary sequence is the estimated ratio η of the number of non-zero elements to the sequence unit length [32], [41]. Based on the definition of $\{\boldsymbol{u}_r^{(m,k)}\}$, the performance η of the proposed sequence is $\frac{\ell_h}{\ell_h+1}\eta_p$, where η_p denotes the η of the binary/ternary perfect sequence $\boldsymbol{p}$.

When the sequences of a zcz sequence set have many zero elements, we can easily construct a zcz sequence set having higher (close to one) ϵ [4]. However, the application of a sequence with many zero elements will have a low signal-to-noise ratio (S/N). Therefore, we are required to make the zero elements of the sequences as few as possible [24]. For this purpose, we must consider both parameters ϵ and η for the performance evaluation of a sequence set by using their product, $\epsilon\eta \le 1$ [24]. This product is $\left(\frac{\ell_h}{\ell_h+1}\right)^2\eta_p$, which is approximately equal to η_p, the η of the binary/ternary perfect sequence $\boldsymbol{p}$, for sufficiently large ℓ_h.

From Theorem 2, the proposed sequence set $\{\boldsymbol{u}_r^{(m,k)}\}$ has a zcz for the aperiodic correlation functions, the even correlation function, and the odd correlation function, simultaneously. The spreading sequence for an AS-CDMA (QS-CDMA) system should have a zcz for both the even and odd correlation functions, as in the case of the proposed sequence [11], [30], [32], [40], [41]. The proposed sequence set can be applied to AS-CDMA in the same manner as discussed in several previous studies [11], [30], [32], [40], [41]. Assigning the subset of the proposed sequence set to the cells of CDMA systems can reduce the interference between the nodes of the different cells more efficiently than can the existing zcz sequences.

4.2 Structure of zcz Width Related to the Subsets of the Proposed Sequence Set

The proposed sequence set $\{\boldsymbol{u}_r^{(m,i)}|0 \le i < \ell_p,\ 0 \le r < 2\ell_h\}$ has ℓ_p subsets of size $2\ell_h$. The j-th subset is $\{\boldsymbol{u}_r^{(m,j)}|0 \le r < 2\ell_h\}$.

The distance between the proposed sequence pair $\boldsymbol{u}_r^{(m,j)}$ and $\boldsymbol{u}_s^{(m,k)}$ is denoted by $\Lambda(j,k)$, which is equal to the difference $|j-k|$.

From Eqs. (28a)-(28e), and (32), we have the following theorem:

Theorem 3:

For $(j,\ r) = (k,\ s), 0 < |\tau| \le 2^{m+1} - 1$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,j)},u_s^{(m,k)}}(\tau) = \overset{\mathrm{A}}{\theta}_{u_r^{(m,j)},u_s^{(m,k)}}(\tau) = 0. \quad (34a)$$

For $(j,\ r) \ne (k,\ s), 0 \le |\tau| \le 2^{m+1} - 1$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,j)},u_s^{(m,k)}}(\tau) = \overset{\mathrm{A}}{\theta}_{u_r^{(m,j)},u_s^{(m,k)}}(\tau) = 0. \quad (34b)$$

For $1 < |j-k| < \ell_p - 1, |\tau| \le 2^{m+1}(\ell_w\,|j-k| - 1)$,

$$\overset{\mathrm{P}}{\theta}_{u_r^{(m,j)},u_s^{(m,k)}}(\tau) = 0. \quad (34c)$$

The zcz of sequence pair $\boldsymbol{u}_r^{(m,j)}$ and $\boldsymbol{u}_s^{(m,k)}$ for $1 < |j-k| < \ell_p - 1$ is wider than that common to all the sequences of the sequence set $\{\boldsymbol{u}_r^{(m,j)}\}$.

For simplicity, we denote the zcz of sequence pair $\boldsymbol{u}_r^{(m,j)}$ and $\boldsymbol{u}_s^{(m,k)}$ by $\mathcal{Z}^{(m)}(j,k;\ r,s)$. Theorem 3 shows the following: For a longer distance $|j-k|$ between the corresponding subsets $\boldsymbol{U}^{(m,j)}$ and $\boldsymbol{U}^{(m,k)}$ for $1 < |j-k| < \ell_p - 1$, the correlation function of phase shift τ of the sequences of $\boldsymbol{u}_r^{(m,j)} \in \boldsymbol{U}^{(m,j)}$ and $\boldsymbol{u}_s^{(m,k)} \in \boldsymbol{U}^{(m,k)}$ has a wider zcz ($|\tau| \le 2^{m+1}(\ell_w\,|j-k| - 1)$). This width of the zcz of a pair of the proposed sequences enables flexible design in applications of the proposed sequence set.

The proposed sequence set can be generated from any given binary/ternary perfect sequence and an Hadamard matrix. There is no limitation of the length of the perfect sequence or the order of the Hadamard matrix for the generation of the proposed sequence set.

Here, we discuss the applications of the proposed sequence set that is generated from a perfect sequence of length 4 or 7. For the case of a perfect sequence $\boldsymbol{p}$ of length 7 ($\ell_p = 7$), the proposed sequence set can be applied to a wireless LAN system having its access points on a common circle as shown in Fig. 1. In Fig. 1, $\boldsymbol{U}^{(m,j)}$ for $0 \le j < 7$ are

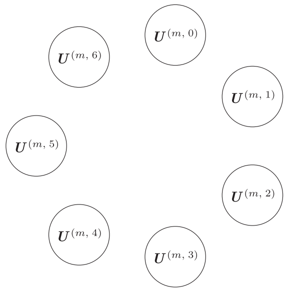

Fig. 1 Wireless LAN access points on a common circle for $\ell_p = 7$.

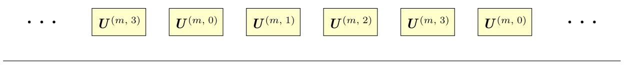

Fig. 2 Ultrasonic transmit-receive elements equally spaced on a line for $\ell_p = 4$.

sequentially assigned to the wireless LAN access points. For the case of $\boldsymbol{p}$ of length 4 ($\ell_p = 4$), the proposed sequence set can be applied to a system having ultrasonic transmit-receive elements located at equal intervals on a common line as shown in Fig. 2. Each rectangle in Fig. 2 is an ultrasonic transmit-receive element. The notation in the rectangles and the circles in Fig. 1 indicates the subset $\boldsymbol{U}^{(m,j)}$ which is assigned to the transmit-receive element. The $\ell_p = 4$ and 7 cases for Figs. 1 and 2, respectively, can similarly be constructed. Please note that the length of the perfect sequence ℓ_p is not restricted to 4 or 7.

In Fig. 2, one of $\boldsymbol{U}^{(m,j)}$ for $0 \le j < 4$ is assigned to each transmit-receive element in sequence.

5. Applications of the Proposed Sequence Set

The correlation properties of the proposed sequence can be utilized for various applications, including signal detection, synchronization, signal delay detection, and channel separation. The proposed sequence set can also improve the performance of radar systems. Assigning the sequences of the proposed sequence set to the transmit-receive elements of a synthetic-aperture ultrasonic imaging system [17] can improve the S/N of the obtained image. We also evaluate the performance of the application of the proposed sequence set for radar pulse compression and ultrasonic imaging.

5.1 Ultrasonic Imaging Using the Proposed Sequence Set

In this section, we consider ultrasonic synthetic-aperture focusing techniques with a zcz sequence set, as well as techniques with a Walsh sequence [33] and a zcz sequence set [15], [17]. Ultrasonic waves are simultaneously transmitted by transmitters and reflected by the observed objects. The reflected ultrasonic waves are detected by receivers, which convert the waves into digital signals, and the correlation function of the digital signal and the transmitted signal (reference signal) is calculated. The phase shift of the peak of the calculated correlation function is used to determine the delay of the detected wave. The delay indicates the length of the path from the transmitter to the object and then to the receiver. To compare the performance of ultrasonic imaging using the proposed sequence set with the performance using a zcz sequence set constructed by the previous method, an ideal environment is simulated.

Because M-sequences have good low-correlation properties, they are used in various applications [2], [9]. However, low-correlation properties do not work well in applications with superimposed signals having various levels of power; in these cases, the zcz property is preferable [15], [17]. Various studies have examined ultrasonic synthetic-aperture focusing techniques for sequences with particular properties [1], [10], [15], [25], [33], [34], [53].

To evaluate the performance of the proposed sequence, we simulated ultrasonic synthetic-aperture imaging that uses the proposed sequence set. To focus on a target image, synthetic-aperture imaging estimates the distance from the ultrasonic transmitter to the target and the distance from the target to the ultrasonic detector, as shown in Fig. 3 [8].

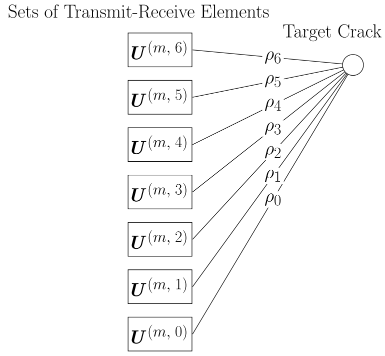

Fig. 3 Geometric relation between the transmit-receive elements and the focal point.

5.2 Effect of the Proposed Sequence Set

Figure 3 demonstrates how the proposed sequence set can be used for ultrasonic synthetic-aperture imaging. The circle in the figure indicates the crack to be detected. The rectangles indicate the sets of ultrasonic receive-transmit elements. A set of ultrasonic receive-transmit elements can realize a set of receive-transmit elements. The sequences $\boldsymbol{u}_r^{(m,k)}$ of the k-th subset $\boldsymbol{U}^{(m,j)}$ are assigned to the k-th set of receive-transmit elements. The symbol in the rectangle indicates the subset assigned to the set of receive-transmit elements. In Fig. 3, the k-th sets of receive-transmit elements for $0 \le k < 7$ are illustrated. The zcz for the sequence pair $\boldsymbol{u}_r^{(m,k)}$ and $\boldsymbol{u}_{r'}^{(m,k')}$ is used to synchronize with a particular signal and eliminate interference from other signals. Therefore, the zcz of a pair of sequences which are assigned to the elements at a longer distance must be wider than the zcz of a pair of sequences which are assigned to the elements at a shorter distance. Therefore, the zcz of the sequence pair $\boldsymbol{u}_r^{(m,0)}$ and $\boldsymbol{u}_{r'}^{(m,3)}$ is designed to be wider than the zcz of the sequence pair $\boldsymbol{u}_r^{(m,0)}$

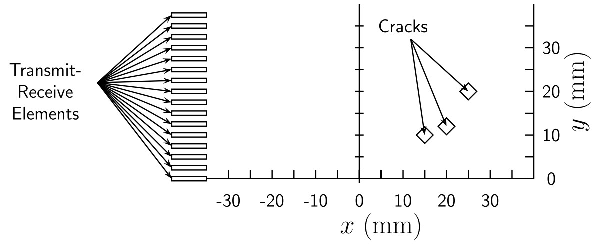

Fig. 4 Simulated environment.

and $\boldsymbol{u}_{r'}^{(m,1)}$.

The line segment in the figure indicates the path from each set of receive-transmit elements to the target crack.

In Fig. 3, when the receiving element (element 5) detects the signal sent by the transmitting element (element 1) in Fig. 3, the round-trip time $\rho_{1,5}$ of the signal is equal to $\rho_1 + \rho_5$. The one-way time ρ_i is equal to r_i/v, where r_i is the distance between the target and transmitter i, and v is the propagation velocity of the ultrasonic wave. Since $|\rho_{1,6} - \rho_{1,1}| = |\rho_6 - \rho_1|$ is larger than $|\rho_{1,6} - \rho_{1,5}| = |\rho_6 - \rho_5|$ and $|\rho_{1,2} - \rho_{1,1}| = |\rho_2 - \rho_1|$, the proposed sequence set with 7 subsets can be applied to a system of receive-transmit elements arranged in 7 sets.

The synthetic-aperture image $S(x, y)$ is obtained as a function of the position (x, y) by computing the convolution of the detected waveform and the reference waveform, which is the detected waveform at a single point located at (x, y) [8]. Details of the computation of synthetic-aperture imaging can be found in [34].

5.3 Performance Evaluation of the Application to Ultrasonic Imaging

To evaluate the performance of the proposed sequence for an ultrasonic imaging application, an ideal environment was simulated, as shown in Fig. 4. In this simulation of ultrasonic imaging of cracks in a concrete medium, the following parameters were used: the number of transmit-receive elements was 16, the wavelength of the ultrasonic wave was 8 mm, the sampling rate was 40 mega-samples per second (40 Msps), the number of ultrasonic waves per one element of the sequence was 4, and the propagation velocity of the ultrasonic wave was 6 km/s. We used the proposed sequence set of length 160 and an M-sequence of length 163. The proposed sequence is constructed from an Hadamard matrix of order $\ell_h = 4$, $\ell_p = 4$, and $m = 1$; the length of the sequence is $L^{(1)} = 2^3(4 + 1) \times 4 = 160$.

A comparison of the performance of the ultrasonic imaging using the proposed sequence set with that using an M-sequence of length 163 and the simulated ideal environment is shown in Fig. 4. The coordinates of the three cracks in Fig. 4 are (22.5 mm, 10.0 mm), (25.0 mm, 12.0 mm), and (27.5 mm, 20.0 mm). Figure 5 shows an image constructed by using the proposed sequence set, each sequence of length 160, and Fig. 6 shows an image constructed by using the M-sequence. The intensity of each image is normalized by

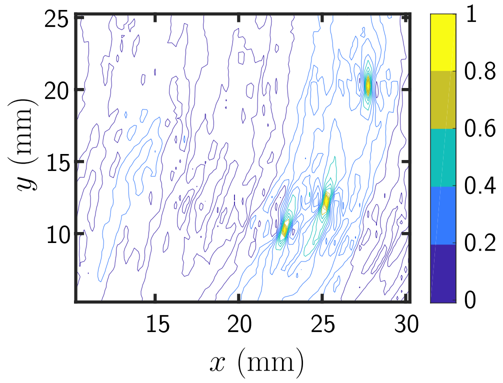

Fig. 5 Image constructed by using the proposed sequence set with a sensor array.

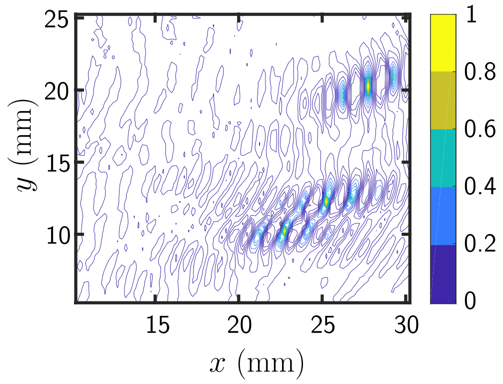

Fig. 6 Image constructed by using an M-sequence with a sensor array.

Table 1 Detected peaks in Figs. 5 and 6.

	Coordinates of the detected peaks (mm)			
Proposed sequence set (Fig. 5)	(27.5, 20.0)	(25.0, 12.0)	(22.5, 10.0)	Correctly detected
M-sequence (Fig. 6)	(27.5, 20.0)	(25.0, 12.0)	(22.5, 10.0)	Correctly detected
	(24.0, 20.8)	(21.0, 19.5)	(26.5, 12.3)	Incorrectly detected
	(23.5, 11.8)	(21.0, 9.8)	(24.0, 10.0)	

setting the peak intensity to 1. The color bars indicate the intensities of the contour curves in the figures. The simulated results shown in Figs. 5 and 6 indicate that the proposed sequence set is better than a zcz sequence set constructed by the previous method. In particular, Fig. 5 displays an image with three clear peaks, which correspond to three target cracks and are clear. The coordinates of the peaks detected in Figs. 5 and 6 are listed in Table 1. The coordinates of the three peaks in Fig. 5 are (22.5 mm, 10.0 mm), (25 mm, 12.0 mm), and (27.5 mm, 20.0 mm); all the peaks are identical, with respect to their coordinates, to the original cracks. In contrast, we can find nine peaks in Fig. 6. Three of the nine peaks have the same coordinates as the original cracks shown in Fig. 5. The image constructed by the M-sequence has six artifact peaks at (24.0 mm, 20.8 mm), (21.0 mm, 19.5 mm), (26.5 mm, 12.3 mm), (23.5 mm, 11.8 mm), (21.0 mm, 9.8 mm), and (24.0 mm, 10.0 mm), as shown in Fig. 6. These six peaks constitute noise with respect to crack detection.

5.4 Visible Light Communication

Multi-user visible light communication (VLC) requires a channel separation mechanism [7], [38], [39]. Although VLC uses binary ('on' and 'off') codes, a ternary sequence set can be applied to multi-color or multi-wavelength light. There exist various reports on multi-color or multi-wavelength VLC [37], [60], [62]. Almost all existing VLC systems are designed to be used without any signal delay. However, it is not easy to synchronize the clocks of all the devices for a VLC system. Therefore, it is necessary to consider the differences in the clocks of the communication nodes in an actual system. By applying the proposed binary zcz sequence set to a VLC system, the system can function properly despite a clock difference within the time which is associated with the zcz of the sequence set.

6. Conclusions

A new construction scheme for a ternary zcz sequence set was presented. The proposed zcz sequence set can be generated from an arbitrary pair of an Hadamard matrix of order ℓ_h and a binary or ternary perfect sequence of length ℓ_p for an arbitrary non-negative parameter $m \geq 0$.

For $m \geq 0$, the sequence set of order $(m+1)$ is constructed from the sequence set of order m by sequence concatenation and interleaving. The sequence set has ℓ_p subsets of size $2\ell_h$.

The proposed sequence set is $\boldsymbol{U}^{(m)}$ for $m \geq 0$, which consists of ℓ_p subsets of $2\ell_h$ sequences, each of length $2^{m+2}(\ell_h+1)\ell_p$. The periodic correlation function and the aperiodic correlation function of a pair of the proposed sequence sets $\boldsymbol{u}_r^{(m,j)}$ and $\boldsymbol{u}_s^{(m,k)}$ satisfy the following:

$$\begin{aligned}
&\text{For } (j,\ r) = (k,\ s), 0 < |\tau| \leq 2^{m+1}-1,\\
&\quad \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(m,j)},\boldsymbol{u}_s^{(m,k)}}(\tau) = \overset{\mathrm{A}}{\theta}_{\boldsymbol{u}_r^{(m,j)},\boldsymbol{u}_s^{(m,k)}}(\tau) = 0.\\
&\text{For } (j,\ r) \neq (k,\ s), 0 \leq |\tau| \leq 2^{m+1}-1,\\
&\quad \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(m,j)},\boldsymbol{u}_s^{(m,k)}}(\tau) = \overset{\mathrm{A}}{\theta}_{\boldsymbol{u}_r^{(m,j)},\boldsymbol{u}_s^{(m,k)}}(\tau) = 0.\\
&\text{For } 1 < |j-k| < \ell_p - 1, |\tau| \leq 2^{m+1}(\ell_w\,|j-k|-1),\\
&\quad \overset{\mathrm{P}}{\theta}_{\boldsymbol{u}_r^{(m,j)},\boldsymbol{u}_s^{(m,k)}}(\tau) = 0.
\end{aligned}$$

The sequence set $\{\boldsymbol{u}_r^{(m,k)}\}$ is $Z(2^{m+2}(\ell_h+1)\ell_p,\ 2\ell_h\ell_p,\ 2^{m+1}-1)$. The zcz of sequence pair $\boldsymbol{u}_r^{(m,j)}$ and $\boldsymbol{u}_s^{(m,k)}$ for $1 < |j-k| < \ell_p - 1$ is wider than that common to all the sequences of the sequence set $\{\boldsymbol{u}_r^{(m,j)}\}$. Such a greater width in the latter case enables flexible design of applications of the proposed sequence set. The proposed sequence is suitable for a heterogeneous wireless network, which is one of the candidates for the fifth-generation mobile networks.

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Takafumi Hayashi received his B.E., M.E., and Ph.D. degrees in applied physics from the University of Tokyo, Tokyo, Japan, in 1985, 1987, and 1992, respectively. From 1989 to 1992, he was a Research Associate at the University of Tokyo. From 1993 to 2004, he was an Associate Professor at the University of Aizu. From 2004 to 2016, he was a Professor at the University of Aizu. From 2016, he has been a Professor at Niigata University. His current research interests include sequence design, signal processing, image analysis, mobile communications, intelligent infrastructure, messaging network, secure cloud computing, smart grids, and information security management. Dr. Hayashi is a member of the ACM, IET, JSIAM, and SPIE. He is a senior member of the IEICE and IEEE.

Takao Maeda received his B.S. and M.S. degrees in mathematics from the University of Tokyo and Hokkaido University, Japan, in 1980 and 1982, respectively, and his Ph.D. degree in computer science and engineering from the University of Aizu in 2013. From 1982 to 1994, he was a Researcher at the Systems Development Laboratory of Hitachi Limited. Since 1994, he has been an Assistant Professor at the University of Aizu. His research interests include sequence design, number theory, and the mathematical science of puzzles, games, and so on. Dr. Maeda is a member of the IEICE, IPSJ, JSIAM, and JSSST.

Anh T. Pham received his B.E. and M.E. degrees, both in electronics engineering, from the Hanoi University of Technology, Vietnam in 1997 and 2000, respectively, and his Ph.D. degree in information and mathematical sciences from Saitama University, Japan in 2005. From 1998 to 2002, he was with NTT Corporation in Vietnam. Since April 2005, he has been on the faculty of the University of Aizu, where he is currently a Professor of Computer Communications at the School of Computer Science and Engineering. Dr. Pham also holds an Adjunct Professor position at the Vietnam National University/University of Engineering and Technology. His research interests are in the broad areas of communication theory and networking with a particular emphasis on modeling, design, and performance evaluation of wired/wireless communication systems and networks. Dr. Pham received a Japanese government scholarship (MonbuKagaku-sho) for his Ph.D. study. He also received a Vietnamese government scholarship for his undergraduate study. Dr. Pham is a senior member of the IEEE. He is also a member of the IEICE and OSA.

Shinya Matsufuji received his B.E. from Fukuoka University, Japan, in 1977, and his Dr.Eng. degree in computer science and communication engineering from Kyushu University, Fukuoka, Japan, in 1993. From 1984 to 2002, he was a Research Associate at the Department of Information Science, Saga University. From 2002 to 2012, he was an Associate Professor at the Department of Computer Science and Systems Engineering, Yamaguchi University. He is currently a Professor at the Graduate School of Science and Engineering, Yamaguchi University. His current research interests include sequence design and spread-spectrum systems. Dr. Matsufuji is a member of the IEEE and IEICE.