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Redundancy eliminations for a polynomial solution of the graph Hamiltonicity problem

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Supplementary Notes: abbreviations used in the report: **pp-solution** - possible partial solutions; **B-edge** – boundary edge, **B-case** – **boundary case, B-communicator** – boundary communicator, **n-invariant** – node invariant, **r-invariant** – regional invariant, **RRI** - redundancy related index, **RUC** - region under consideration, **NR** – neighboring region, **RE** – redundancy elimination, **BR** – basic region, **TCs** - tree chains

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Redundancy eliminations for a polynomial solution of the graph Hamiltonicity problem

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Abstract. A regional invariant method based on redundancy eliminations is presented for the polynomial solution of the graph Hamiltonicity problem. This method includes searching logarithmic size regions, considering all possible partial solutions (pp-solutions) within corresponding regions and removing redundant pp-solutions. The pp-solutions preserved are employed for extracting edge value invariants in the regions and composing a system of linear algebraic equations which is solved by a standard method applied for systems with symbolic parameters. Within the method, regional redundancy and redundancy with remote impact eliminations are applied in order to split the problem into a polynomial number of tractable subproblems.

Keywords and phrases: regional redundancy, redundancy with a remote impact, regional invariant, symbolic parameter system, Hamiltonicity polynomial solution, P=NP.

1. Preliminary considerations

One of the main ideas behind the approach is the regional existence of possible partial solutions (pp-solutions) that can be removed without altering Hamiltonicity. Another idea is the regional existence of invariants which are represented by values of some edges in regions and do not include values of edges from the "external world" (though for discovering such invariants, values of edges from the external world can be employed). The invariants are extracted within eliminating the redundant pp-solutions and within splitting the pp-solution sets across all regions. The efficiency of this process is based on reducing the problem to a polynomial number of the subproblems that are related to different slices of pp-solutions and solved as the systems of linear algebraic equations. To demonstrate this approach, we select the Hamiltonicity problem on a planar graph ***G* = *G* (**V**,** E**)**, where V is a set of nodes and E is a set of edges. It is assumed that the graph is undirected with no weighted edges. It is also assumed that the number of nodes is ***N***, the number of edges is ***M*** and the node degree is **3.** The problem of searching a Hamiltonian cycle in such a graph is still NP-complete [1]. Our goal is to present a polynomial algorithm for finding a Hamiltonian cycle or to show the graph is non-Hamiltonian.

For simplicity, we assume that the graph is 3-edge connected (otherwise, our initial problem is reduced to problems of smaller sizes). In addition, nodes of 3-degree faces are replaced by one node [2] and the problem is reduced to a graph with a smaller number of nodes. In any case, we expect that the graph is big enough.

Two symbolic parameters **H** and **F** are introduced to represent values of edges: E_1 **= H** means that edge E_1 belongs to a Hamiltonian cycle and E_2 **= F** means that edge E_2 does not belong to a Hamiltonian cycle (**F**ailed to be a part of a cycle). For edges E_1, E_2 and E_3 incident to a common node, we can write a **node invariant** (**n**-invariant) as the following equation:

$$E_1 + E_2 + E_3 = \mathbf{2H} + \mathbf{F}, \quad (1.1)$$

representing a necessary (for the graph Hamiltonicity) feature in this node. This equation has to be valid in all nodes.

In a sense, **H** and **F** represents "***yes***" and "***no***" (1 and 0), but with different "units of measure." For such parameters, symbolic expression

$$E_4 + E_5 + E_6 + E_7 + E_8 = 5\mathbf{H}$$

means that all E_i (i=4, …, 8) are equal to **H,** expression $E_1 = 2\mathbf{H} - \mathbf{z_1}$ is valid only if $\mathbf{z_1} = \mathbf{H}$ and expressions $E_1 = (\mathbf{H} + \mathbf{F})/2$ or $E_1 = 2\mathbf{H} - \mathbf{F}$ are just invalid. **H** and **F** are convenient for reasoning about the symbolic expressions of edge values.

Edges of a cut-set separating a subgraph (region) from other parts of the graph are called boundary (**B**-) **edges** of a **B-set**. A set of B-edge values is called a **B-set value**. The B-edges are arranged in a set of pairs each of which represents an **in-out link** related to a Hamiltonian cycle path going through the region. Paths of different pairs are not intersected and jointly cover all regional nodes.

This set of B-edge pairs (corresponding to their in-out links) is called a **B-communicator**. For a B-set value, there can be a number of different B-communicators. Together a B-set value and a corresponding B-communicator are called a **B-case**. In other words, the number of B-cases with the same B-set value is equal to the number of different B-communicators.

Fig.1 depicts examples of B-edges, B-set values, in-out link sets and B-cases:

- B_1, B_2, B_3, and B_4 are B-edges in all examples;

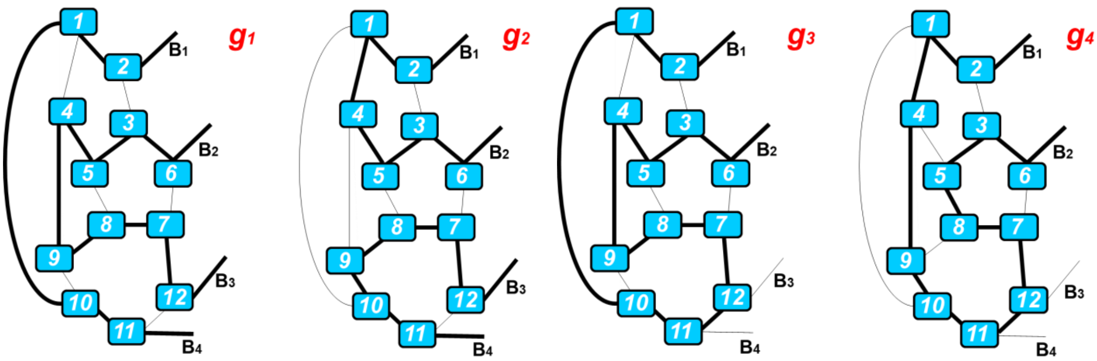


Fig. 1. Examples related to region boundary edges

- (**H,H,H,H**) is a B-set value in examples g_1 and g_2; (**H,H,F,F**) is a B-set value in examples g_3 and g_4;

- {(B_1, B_4), (B_2, B_3)} is a B-communicator in example g_1, {(B_1, B_2), (B_3, B_4)} is a B-communicator in example g_2. In other words, examples g_1 and g_2 present two different B-cases for the same B-set value.

- Finally, {(B_1, B_2)} is a B-communicator (of one in-out link) in example g_3; it is also a B-communicator in example g_4. The coincidence of the B-communicators means that example g_3 and g_4 present one B-case and any of two different paths inside the region is allowed to be used.

A linear algebraic equation related to some region edges (for example, $E_i + E_j + E_k$ **=** 2**H** + **F,** $E_i + E_j$ **= H** + **F,** E_i = **H** or $E_i - E_j$ **=** 0) which is valid for all possible B-cases is called a regional invariant.

Such invariants are extracted in logarithmic-size regions where the numbers of region B-edges and the number of interior faces are ≤ $c \times log_2 N$ (***c*** is a pre-defined constant). In order to solve a Hamiltonicity problem, we need ***N***/2 **regional invariants** (**r**-invariants), which have to be independent of **n**-invariants (1.1). In order to check independence, we can calculate the rank of a corresponding matrix. **R**-invariants together with the **n**-invariants allow composing a system of ***M*** linear algebraic equations (***M*** is the edge number), which is solved by a standard method (Gaussian elimination) applied for **symbolic parameter systems** [3]. Absence of an appropriate solution means that the graph is non-Hamiltonian.

In the next sections, we consider ideas behind our approach: how to apply a system of linear algebraic equations with symbolic parameters for discovering pp-solutions, what redundant pp-solutions are and how to extract **r**-invariants. Special attention is paid to regional redundancy and redundancy with a remote impact, as well as, covering the graph with a tree of region chains and creating a set of the subproblems based on splitting the pp-solutions across all regions.

2. Valid pp-solutions and redundancy eliminations

Our eliminations are based on redundancies: they are regional pp-solutions (or chains of pp-solutions in chains of regions) that can be removed without altering the graph Hamiltonicity.

In order to apply the redundancy eliminations, first we have to find all valid (feasible) pp-solutions in the region under consideration.

A pp-solution is defined as a B-communicator set of Hamiltonian non-intersected paths covering jointly all regional nodes, satisfying n-invariants (1.1) in each node and excluding internal cycles of H-valued edges in the region.

This definition implicitly includes another necessary condition of Hamiltonicity which should be applied for feasible pp-solutions:

For any region the symbolic expression of the sum of B-edges (related to any Hamiltonian solution) has to include an even number of H and this number should be ≥ 2.

For each B-set value, its own set of the feasible pp-solutions is discovered. These pp-solutions are divided into clusters of equivalent B-communicators possessing the same set of B-edge pairs. Redundancy elimination is applied to these clusters; it preserves only one representative pp-solution from each cluster of the feasible pp-solutions. In other words, the sum of different B-cases of each B-set value defines the number of pp-solutions left in corresponding regions for extracting the **r**-invariants. The pp-solutions eliminated in such a way are called **redundant pp-solutions**.

Here we consider examples to clarify some aspects of the feasible pp-solutions and redundancy eliminations.

Discovering feasible pp-solutions is based on composing a system of the equations and solving it. Assume that graph ***G*** of Fig. 2 is divided in two regions by edges of a cut-set, which is behind the C-D-H-face path.

Then for the left region edges of ***G,*** we can write the following equations of the (1.1) type:

$$
\begin{aligned}
&E_1 + E_2 + E_3 = 2\mathbf{H} + \mathbf{F} \qquad && E_3 + E_7 + \mathbf{E_{22}} = 2\mathbf{H} + \mathbf{F} \\
&E_2 + E_4 + E_5 = 2\mathbf{H} + \mathbf{F} && E_6 + E_7 + \mathbf{E_{12}} = 2\mathbf{H} + \mathbf{F} \\
&E_4 + E_9 + E_{10} = 2\mathbf{H} + \mathbf{F} && E_8 + E_9 + \mathbf{E_{13}} = 2\mathbf{H} + \mathbf{F} \\
&E_5 + E_6 + E_8 = 2\mathbf{H} + \mathbf{F} && E_1 + E_{10} + \mathbf{E_{14}} = 2\mathbf{H} + \mathbf{F},
\end{aligned}
\qquad (2.1)
$$

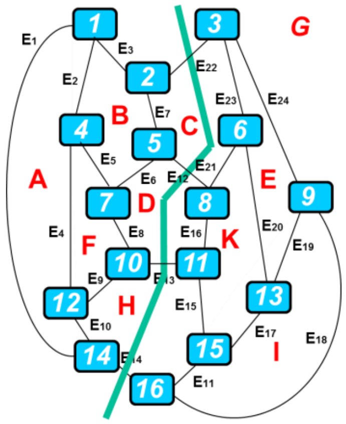


Fig. 2. A cut-set line dividing graph *G* into two regions

where the right column of the equations is related to the B-set values (values of $\mathbf{E_{22}}$, $\mathbf{E_{12}}$, $\mathbf{E_{13}}$, and $\mathbf{E_{14}}$). Since we consider the system of the equations for each possible B-set value (that is for assigned values of $\mathbf{E_{22}}$, $\mathbf{E_{12}}$, $\mathbf{E_{13}}$, and $\mathbf{E_{14}}$), then there are ten unknowns in eight equations. To decrease the number of B-cases to consider, we take into account that B-edge values, making a contribution in such a B-set value, satisfy the above mentioned condition of Hamiltonicity:

$$E_{22} + E_{12} + E_{13} + E_{14} = 2\mathbf{H} + 2\mathbf{F} \text{ or}$$
$$E_{22} + E_{12} + E_{13} + E_{14} = 4\mathbf{H}. \qquad (2.2)$$

As an example, let $E_{22} = E_{12} = \mathbf{F}$ and $E_{13} = E_{14} = \mathbf{H}$. Solving system (2.1) gives us a solution for eight edges depending on **H**, **F** and two symbolic parameters E_2 and E_5:

$$E_1 = \mathbf{H}+\mathbf{F}- E_2,\ E_3 = \mathbf{H},\ E_4 = 2\mathbf{H}+\mathbf{F}- E_5- E_2,\ E_6 = \mathbf{H},$$
$$E_7 = \mathbf{H},\ E_8 = \mathbf{H}+\mathbf{F}- E_5,\ E_9 = E_5,\ E_{10} = E_2.$$

Possible values of $E_2 = \mathbf{H}$, $E_5 = \mathbf{F}$ and $E_2 = \mathbf{F}$, $E_5 = \mathbf{H}$ give us two pp-solutions:

$$E_{14} = E_{10} = E_4 = E_2 = E_3 = E_7 = E_6 = E_8 = E_{13} = \mathbf{H} \text{ and}$$
$$E_{14} = E_1 = E_3 = E_7 = E_6 = E_5 = E_4 = E_9 = E_{13} = \mathbf{H}$$

(in both pp-solutions other edge values are equal to **F**), while values of $E_2 = \mathbf{H}$ and $E_5 = \mathbf{H}$ give a pp-solution, which is eliminated because of a subtour of an **H**-valued edge cycle $\{E_2 = \mathbf{H}, E_3 = \mathbf{H}, E_7 = \mathbf{H}, E_6 = \mathbf{H}, E_5 = \mathbf{H}\}$. It is important to note that in the feasible pp-solutions, the pair of $E_{13} = E_{14} = \mathbf{H}$ has the same B-communicator (entering the region at E_{14} and leaving it at E_{13}, or vice versa).

This means that if one of these pp-solutions is a part of a Hamiltonian cycle then the other pp-solution is also a part of another Hamiltonian cycle which differs from the former one only in our region.

For the second example with E_{22} = E_{13} = **F** and E_{12} = E_{14} = **H,** edges E_2 and E_5 can also play a role as symbolic parameters. For any possible variation of them, there is only one feasible pp-solution:

$$E_{14} = E_{10} = E_9 = E_8 = E_5 = E_2 = E_3 = E_7 = E_{12} = \mathbf{H}$$

and for the third example with E_{22} = E_{12} = **H** and E_{13} = E_{14} = **F,** there are no pp-solutions compatible with the necessary conditions of Hamiltonicity; one is because of an **H**-valued edge cycle and two others are because of invalid (infeasible) edge values in the form of 2**H**-**F**.

Finally, let us consider one more example with E_{22} = E_{12} = E_{13} = E_{14} = **H.** There are three pp-solutions (of two paths each):

$$E_{14} = E_{10} = E_9 = E_{13} = \mathbf{H};\ \ E_{12} = E_6 = E_5 = E_2 = E_3 = E_{22} = \mathbf{H}$$
$$E_{14} = E_{10} = E_4 = E_2 = E_3 = E_{22} = \mathbf{H};\ \ E_{13} = E_8 = E_6 = E_{12} = \mathbf{H}$$
$$E_{14} = E_1 = E_3 = E_{22} = \mathbf{H};\ \ E_{13} = E_9 = E_4 = E_5 = E_6 = E_{12} = \mathbf{H}.$$

In the first pp-solution, the in-out pairs of B-communicator are (E_{14}, E_{13}) and (E_{12}, E_{22}), while in the second one, they are (E_{14}, E_{22}) and (E_{13}, E_{12}). This means that if one of these pp-solutions is a part of a Hamiltonian cycle, then it is not necessarily the same for the other pp-solution. On the other hand, in both the second and third pp-solutions the in-out pairs of B-communicators are (E_{14}, E_{22}) and (E_{13}, E_{12}). This means that one of these pp-solutions can be removed without eliminating the possibility of discovering a possible Hamiltonian cycle.

Now let us consider another example related to the left region of Fig.3.

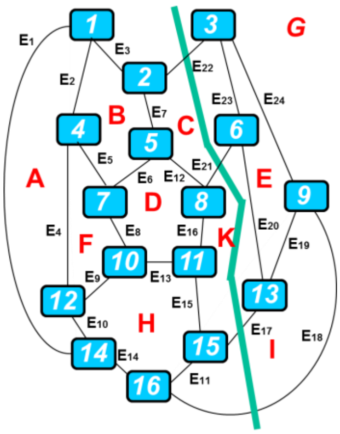


Fig. 3. A cut-set line based on the C-K-I-face path

The system of equations of the (1.1) type is the following:

$$
\begin{aligned}
&E_1 + E_2 + E_3 = 2\mathbf{H} + \mathbf{F} &\quad& E_3 + E_7 + \mathbf{E_{22}} = 2\mathbf{H} + \mathbf{F} \\
&E_2 + E_4 + E_5 = 2\mathbf{H} + \mathbf{F} && E_{12} + E_{16} + \mathbf{E_{21}} = 2\mathbf{H} + \mathbf{F} \\
&E_4 + E_9 + E_{10} = 2\mathbf{H} + \mathbf{F} && E_{11} + E_{15} + \mathbf{E_{17}} = 2\mathbf{H} + \mathbf{F} \qquad (2.3) \\
&E_5 + E_6 + E_8 = 2\mathbf{H} + \mathbf{F} && E_{11} + E_{14} + \mathbf{E_{18}} = 2\mathbf{H} + \mathbf{F} \\
&E_6 + E_7 + E_{12} = 2\mathbf{H} + \mathbf{F} && E_1 + E_{10} + E_{14} = 2\mathbf{H} + \mathbf{F} \\
&E_8 + E_9 + E_{13} = 2\mathbf{H} + \mathbf{F} && E_{13} + E_{15} + E_{16} = 2\mathbf{H} + \mathbf{F}.
\end{aligned}
$$

The B-set values are sets of values of $\mathbf{E_{22}}$, $\mathbf{E_{21}}$, $\mathbf{E_{17}}$, and $\mathbf{E_{18}}$. Taking the names of these edges as symbolic parameters and a solving system (2.3) (with 12 equations and 16 unknowns), we find four independent edges (E_9, E_{10}, E_{11} and E_{16}) allowing all region edges to be presented as symbolic expressions depending on these edges (as well as on E_{22}, E_{21}, E_{17}, and E_{18}**)**. In other words, by considering possible value variations of E_{22}, E_{21}, E_{17}, and E_{18}, as well as E_9, E_{10}, E_{11} and E_{16}, we find all possible pp-solutions in the region. Logarithmic sizes allow applying brute force for this consideration; however, a more elegant approach can also be taken. For example, because of the condition for Hamiltonicity, the B-edge value has to satisfy one of the equalities: $E_{22} + E_{21} + E_{17} + E_{18} = 4\mathbf{H}$ or $E_{22} + E_{21} + E_{17} + E_{18} = 2\mathbf{H} + 2\mathbf{F}$ (in fact, the first equality means that $E_{22} = \mathbf{H}$, $E_{21} = \mathbf{H}$, $E_{17} = \mathbf{H}$ and $E_{18} = \mathbf{H}$**)**. In addition, by consecutively adding equations of $E_9 – E_{11} = 0$ or $E_9 + E_{11} = \mathbf{H}+\mathbf{F}$ types, we can reason about the symbolic expressions of the edge values. Under condition of $E_{22} + E_{21} + E_{17} + E_{18} = 4\mathbf{H}$, let’s consider two assumptions, that $E_9=E_{11}$, $E_{10}=E_{16}$ ($E_9-E_{11}=0$, $E_{10}-E_{16}=0$) and $E_9=E_{11}$, $E_{10}\neq E_{16}$ ($E_9-E_{11}=0$, $E_{10}+E_{16}=H+\mathbf{F}$). For the first assumption, the linear solver [3] provides symbolic expressions for edge values, where we can see $E_1=3\mathbf{H}+\mathbf{F}-E_4-2E_{10}$ and $E_2=-2\mathbf{H}-\mathbf{F}+E_4+3E_{10}$; the first can be valid only with $E_{10}=\mathbf{H}$ and, after that, the second can be valid only for $E_4=\mathbf{F}$ (see some rules for reasoning about edge values in Appendix 1). Applying $E_{10}=\mathbf{H}$ and $E_4=\mathbf{F}$ gives us a pp-solution of two paths (of **H**-valued edges):

$\mathbf{E_{17}}$, E_{11}, $\mathbf{E_{18}}$; $\mathbf{E_{21}}$, E_{16}, E_{13}, E_9, E_{10}, E_1, E_2, E_5, E_6, E_7, $\mathbf{E_{22}}$.

For the second assumption, we receive among others $E_{11}= (E_5+ E_7)/2$ and $E_1= -(2\mathbf{F}-E_5-3E_7)/2$. The first can be valid only if $E_5=E_7$ and the second if $E_5=E_7=\mathbf{F}$. Applying $E_5=\mathbf{F}$ and $E_7=\mathbf{F}$ gives us another pp-solution of two paths:

$\mathbf{E_{17}}$, E_{15}, E_{13}, E_8, E_6, E_{12}, $\mathbf{E_{21}}$**;** $\mathbf{E_{18}}$, E_{14}, E_{10}, E_4, E_2, E_3, $\mathbf{E_{22}}$**.**

Similarly, we can consider assumptions $E_9\neq E_{11}$, $E_{10}=E_{16}$ and $E_9\neq E_{11}$, $E_{10}\neq E_{16}$ and obtain three pp-solutions:

$\mathbf{E_{17}}$, E_{15}, E_{13}, E_9, E_4, E_5, E_6, E_{12}, $\mathbf{E_{21}}$; $\mathbf{E_{18}}$, E_{14}, E_1, E_3, $\mathbf{E_{22}}$**,**
$\mathbf{E_{17}}$, E_{15}, E_{16}, $\mathbf{E_{21}}$**;** $\mathbf{E_{18}}$, E_{14}, E_1, E_2, E_4, E_9, E_8, E_6, E_7, $\mathbf{E_{22}}$,
$\mathbf{E_{17}}$, E_{15}, E_{13}, E_9, E_{10}, E_{14}, $\mathbf{E_{18}}$**;** $\mathbf{E_{21}}$, E_{12}, E_6, E_5, E_2, E_3, $\mathbf{E_{22}}$.

Totally, three pp-solutions have B-communicator {(E_{17}, E_{21}), (E_{18}, E_{22})} and two pp-solutions have B-communicator {(E_{17}, E_{18}), (E_{21}, E_{22})}.

The redundancy elimination follows extracting pp-solutions acceptable in regions by considering all possible B-set values. First, each cluster of pp-solutions, based on the same B-set value and the same B-communicator, is replaced by one pp-solution (a representative one). In our example, just considered, as well as in the first and fourth examples related to the left region of Fig. 2, one-of-two and two-of-three pp-solutions can be removed. Then, a set of B-cases is created. This set includes the cluster representatives that are pp-solutions based on different B-set values and pp-solutions based on the same B-set value, but with different B-communicators. In general, the number of different B-communicators for a B-set value is less or equal to $3^{\mathbf{p}-1}/2 + 1/2$, where **p** is the number of B-communicator pairs. For example, for a B-set with four, six and eight **H**-valued B-edges, that is for **p**=2, 3 and 4, it is 2, 5 and 14, respectively (see Appendix 2).

The set of B-cases is employed for extracting the **r**-invariants. However, there are alternative ways for pp-solution eliminations; therefore, creating the set of B-cases can be performed within the process of extracting the **r**-invariants or can be postponed until some additional data from neighboring regions becomes available.

3. Examples of extracting regional invariants

R-invariants are extracted from pp-solutions preserved after the redundancy eliminations. In fact, as mentioned above, the eliminations have alternatives in selecting representative pp-solutions; therefore, the final decision about such pp-solutions should be performed within the framework of extracting the **r**-invariants. To demonstrate this, we will continue the consideration of the previous example related to Fig. 3. This includes pp-solutions related to the second equality for B-edges: $E_{22} + E_{21} + E_{17} + E_{18} = 2\mathbf{H} + 2\mathbf{F}$. The number of B-set values is 6:

Table 1: B-set values for the Fig. 3 example

	B_1=E_{22}	B_2=E_{21}	B_3=E_{17}	B_4=E_{18}
1	**H**	**H**	**F**	**F**
2	**H**	**F**	**H**	**F**
3	**H**	**F**	**F**	**H**
4	**F**	**H**	**H**	**F**
5	**F**	**H**	**F**	**H**
6	**F**	**F**	**H**	**H**

For each of these B-set values we consider solutions of the (2.3) system. However, before such a consideration, we can enter symbolic parameters B_j (j=1,2,3,4) into this system and find values of region edges depending on them, as well as on some independent edges (provided by the solver). In this case, they are E_9, E_{10}, E_{11}, and E_{16}. Value variations of this independent edge set (ind-set), applied to each B-set value, allow us to discover all pp-solutions in the region.

As we have already mentioned, logarithmic sizes allow applying brute force for this discovery. However, some reducing operations can be done in advance. For example, four cases with E_9=E_{10}=**F** and any values for E_{11} and E_{16} can be removed in advance from the consideration, because of incompatibility with an n-invariant. A similar thing occurs with E_9=E_{10}=E_{11}=**H** and E_{16}=**F** at B_4=**F;** this is because of an internal H-valued cycle: E_9, E_{10}, E_{14}, E_{11}, E_{15}, E_{13}. This means that if B_4=**F**, for any values of B_1, B_2, and B_3, we should not try cases with E_9=E_{10}=E_{11}=**H** and E_{16}=**F.**

B-set value 1 is related to B_1=B_2=**H**, B_3=B_4=**F.** Then, E_{11} has to be equal only to **H,** and the variation of other independent edges includes five ind-sets:

1: {E_9=E_{10}=E_{16}=**H**}, 2: {E_9=**H**, E_{10}=**F**, E_{16}=**H**}, 3: {E_9=**F**, E_{10}=**H**, E_{16}=**H**},
4: {E_9=**H**, E_{10}=**F**, E_{16}=**F**}, 5: {E_9=**F**, E_{10}=**H**, E_{16}=**F**}.

There are only three feasible pp-solutions for ind-sets 2, 4 and 5:

E_{21}, E_{16}, E_{15}, E_{11}, E_{14}, E_1, E_2, E_4, E_9, E_8, E_6, E_7, **E_{22},**
E_{21}, E_{12}, E_6, E_5, E_4, E_9, E_{13}, E_{15}, E_{11}, E_{14}, E_1, E_3, **E_{22},**
E_{21}, E_{12}, E_6, E_8, E_{13}, E_{15}, E_{11}, E_{14}, E_{10}, E_4, E_2, E_3, **E_{22}.**

They have the same B-communicator {E_{21}, E_{22}}; therefore, the redundancy elimination allows us to preserve only one of these pp-solutions.

B-set value 2 is related to B_1=**H**, B_2=**F**, B_3=**H**, B_4=**F.** Then, E_{11} and E_{16} have to be equal only to **H,** and the variation of other independent edges includes the following ind-sets: 1: {E_9=E_{10}=**H**}, 2: {E_9=**F**, E_{10}=**H**}, 3: {E_9=**H**, E_{10}=**F**}.
There is only one feasible pp-solution for ind-set 1:

E_{17}, E_{11}, E_{14}, E_{10}, E_9, E_{13}, E_{16}, E_{12}, E_6, E_5, E_2, E_3, **E_{22}**.

B-set value 3 is related to B_1=**H**, B_2=**F**, B_3=**F**, B_4=**H.** Then, as above, E_{11} and E_{16} have to be equal only to **H,** and the variation of other independent edges includes the following ind-sets: 1: {E_9=E_{10}=**H**}, 2: {E_9=**F**, E_{10}=**H**}, 3: {E_9=**H**, E_{10}=**F**}. There are no feasible pp-solutions for these ind-sets and B-set value 3 is removed.

B-set value 4 is related to B_1=**F**, B_2=**H**, B_3=**H**, B_4=**F.** Then, as in B-set value 1, E_{11} has to be equal only to **H** and the variation of other independent edges includes five ind-sets:

1: {E_9=E_{10}=E_{16}=**H**}, 2: {E_9=**H**, E_{10}=**F**, E_{16}=**H**}, 3: {E_9=**F**, E_{10}=**H**, E_{16}=**H**},
4: {E_9=**H**, E_{10}=**F**, E_{16}=**F**}, 5: {E_9=**F**, E_{10}=**H**, E_{16}=**F**}.

There are two feasible pp-solutions for ind-sets 2 and 3:

E_{17}, E_{11}, E_{14}, E_1, E_3, E_7, E_6, E_5, E_4, E_9, E_{13}, E_{16}, **E_{21},**
E_{17}, E_{11}, E_{14}, E_{10}, E_4, E_2, E_3, E_7, E_6, E_8, E_{13}, E_{16}, **E_{21}.**

They have the same B-communicator {E_{17}, E_{21}}; therefore the redundancy elimination allows us to preserve only one of these pp-solutions.

B-set value 5 is related to B_1=**F**, B_2=**H**, B_3=**F**, B_4=**H.** Then, E_{11} has to be equal only to **H,** and the variation of other independent edges includes five ind-sets:

1: {E_9=E_{10}=E_{16}=**H**}, 2: {E_9=**H**, E_{10}=**F**, E_{16}=**H**}, 3: {E_9=**F**, E_{10}=**H**, E_{16}=**H**},
4: {E_9=**H**, E_{10}=**F**, E_{16}=**F**}, 5: {E_9=**F**, E_{10}=**H**, E_{16}=**F**}.

There is only one feasible pp-solution for ind-set 5:

E_{18}, E_{11}, E_{15}, E_{13}, E_8, E_5, E_4, E_{10}, E_1, E_3, E_7, E_{12}, **E_{21}.**

B-set value 6 is related to B_1=**F**, B_2=**F**, B_3=**H**, B_4=**H.** Then, E_{16} has to be equal only to **H,** and the variation of other independent edges includes six ind-sets:

1: {E_9=E_{10}=E_{11}=**H**}, 2: {E_9=**H**, E_{10}=**H**, E_{11}=**F**}, 3: {E_9=**H**, E_{10}=**F**, E_{11}=**H**},
4: {E_9=**F**, E_{10}=**H**, E_{11}=**H**}, 5: {E_9=**H**, E_{10}=**F**, E_{11}=**F**}, 6: {E_9=**F**, E_{10}=**H**, E_{11}=**F**}.

There is only one feasible pp-solution for ind-set 2:

E_{17}, E_{15}, E_{16}, E_{12}, E_7, E_3, E_2, E_5, E_8, E_9, E_{10}, E_{14}, **E_{18}.**

In total, thirteen feasible pp-solutions have been discovered:

E_{17}, E_{11}, **E_{18}**; **E_{21}**, E_{16}, E_{13}, E_9, E_{10}, E_1, E_2, E_5, E_6, E_7, **E_{22}**,
E_{17}, E_{15}, E_{13}, E_9, E_{10}, E_{14}, **E_{18};** **E_{21}**, E_{12}, E_6, E_5, E_2, E_3, **E_{22}**,

E_{17}, E_{15}, E_{13}, E_8, E_6, E_{12}, **E_{21};** **E_{18}**, E_{14}, E_{10}, E_4, E_2, E_3, **E_{22},**
E_{17}, E_{15}, E_{13}, E_9, E_4, E_5, E_6, E_{12}, **E_{21}**; **E_{18}**, E_{14}, E_1, E_3, **E_{22},**
E_{17}, E_{15}, E_{16}, **E_{21};** **E_{18}**, E_{14}, E_1, E_2, E_4, E_9, E_8, E_6, E_7, **E_{22},**

E_{21}, E_{16}, E_{15}, E_{11}, E_{14}, E_1, E_2, E_4, E_9, E_8, E_6, E_7, **E_{22},**
E_{21}, E_{12}, E_6, E_5, E_4, E_9, E_{13}, E_{15}, E_{11}, E_{14}, E_1, E_3, **E_{22},**
E_{21}, E_{12}, E_6, E_8, E_{13}, E_{15}, E_{11}, E_{14}, E_{10}, E_4, E_2, E_3, **E_{22},**

E_{17}, E_{11}, E_{14}, E_{10}, E_9, E_{13}, E_{16}, E_{12}, E_6, E_5, E_2, E_3, **E_{22},**

E_{17}, E_{11}, E_{14}, E_1, E_3, E_7, E_6, E_5, E_4, E_9, E_{13}, E_{16}, **E_{21},**
E_{17}, E_{11}, E_{14}, E_{10}, E_4, E_2, E_3, E_7, E_6, E_8, E_{13}, E_{16}, **E_{21},**

E_{18}, E_{11}, E_{15}, E_{13}, E_8, E_5, E_4, E_{10}, E_1, E_3, E_7, E_{12}, **E_{21},**

E_{17}, E_{15}, E_{16}, E_{12}, E_7, E_3, E_2, E_5, E_8, E_9, E_{10}, E_{14}, **E_{18}.**

Seven of these pp-solutions have to be preserved, and six can be eliminated to simplify discovering an **r**-invariant. Extracting values of the independent edges from these pp-solutions gives us Table 2:

Table 2: Values of independent edges

	1	2	3	4	5	6	7	8	9	10	11	12	13
E_9	**H**	**H**	**F**	**H**	**H**	H	H	F	**H**	**H**	**F**	F	**H**
E_{10}	**H**	**H**	**H**	**F**	**F**	F	F	H	**H**	**F**	**H**	H	**H**
E_{11}	**H**	**F**	**F**	**F**	**F**	H	H	H	**H**	**H**	**H**	H	**F**
E_{16}	**H**	**F**	**F**	**F**	**H**	H	F	F	**H**	**H**	**H**	F	**H**

There are four clusters of columns of (1-2), (3-5), (6-8), (10-11), where in each of them, we can preserve only one column, and there are three columns of 9, 12 and 13 which are not removable. Column 9 is coincident with column 1, so cluster (1-2) can be represented by column 1; similarly, column 12 is coincident with column 8 and cluster (6-8) can be represented by column 8. For selecting representatives of clusters (3-5) and (10-11), we should decide about an invariant pattern (E_j=**H**, E_j+E_k=**H**+**F**, E_j=E_k or E_j+E_k+E_p=2**H**+**F**, etc.) to be searched. In our case, pattern E_{10}=**H** leads to columns 3 and 11 as the representatives (see Table 3), while E_9=E_{16} leads to columns 3 and 10 (or 5 and 10) (see Table 4). In other words, we extracted two invariants (E_{10}=**H** and E_9=E_{16}) and one of them can be used in further considerations.

Table 3: Representatives with columns 3 and 11

	1	3	8	9	11	12	13
E_9	H	F	F	H	F	F	H
E_{10}	H	H	H	H	H	H	H
E_{11}	H	F	H	H	H	H	F
E_{16}	H	F	F	H	H	F	H

Table 4: Representatives with columns 3 and 10

	1	3	8	9	10	12	13
E_9	H	F	F	H	H	F	H
E_{10}	H	H	H	H	F	H	H
E_{11}	H	F	H	H	H	H	F
E_{16}	H	F	F	H	H	F	H

To extract new invariants we should consider new regions. Fig. 4 depicts a cut-set line based on the E-K-H-face path and a top-left region we are going to consider. This region is overlapped with the previous one; therefore in the system of equations of (2.3), we add the E_{10}=**H** invariant:

$$
\begin{aligned}
& & E_1 + E_2 + E_3 &= 2\mathbf{H} + \mathbf{F} & \quad E_3 + E_7 + E_{22} &= 2\mathbf{H} + \mathbf{F} \\
& & E_2 + E_4 + E_5 &= 2\mathbf{H} + \mathbf{F} & \quad E_{12} + E_{16} + E_{21} &= 2\mathbf{H} + \mathbf{F} \\
& & E_4 + E_9 + E_{10} &= 2\mathbf{H} + \mathbf{F} & \quad E_{22} + E_{23} + \mathbf{E_{24}} &= 2\mathbf{H} + \mathbf{F} \qquad (3.1) \\
& & E_5 + E_6 + E_8 &= 2\mathbf{H} + \mathbf{F} & \quad E_{21} + E_{23} + \mathbf{E_{20}} &= 2\mathbf{H} + \mathbf{F} \\
& & E_6 + E_7 + E_{12} &= 2\mathbf{H} + \mathbf{F} & \quad E_{13} + E_{16} + \mathbf{E_{15}} &= 2\mathbf{H} + \mathbf{F} \\
E_{10} = \mathbf{H} & & E_8 + E_9 + E_{13} &= 2\mathbf{H} + \mathbf{F} & \quad E_1 + E_{10} + \mathbf{E_{14}} &= 2\mathbf{H} + \mathbf{F}
\end{aligned}
$$

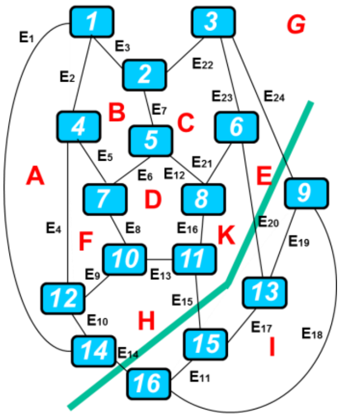


Fig. 4. A cut-set line based on the E-K-H-face path

The B-set values are sets of values of **E_{24}**, **E_{20}**, **E_{15}**, and **E_{14}**. Taking the names of these edges as symbolic parameters and a solving system (3.1) (where 13 equations and 16 unknowns), we find three independent edges (E_4, E_5 and E_{23}). By considering possible value variations of these independent edges, as well as B-edges, we find all possible pp-solutions in this new region.

As for the previous region, the B-edge value has to satisfy one of the equalities: $E_{24} + E_{20} + E_{15} + E_{14} = 4$**H** or $E_{24} + E_{20} + E_{15} + E_{14} = 2$**H** + 2**F.** For $E_{24} + E_{20} + E_{15} + E_{14} = 4$**H**, there are three feasible pp-solutions:

E_{14}, E_{10}, E_9, E_8, E_5, E_2, E_3**,** E_7, E_{12}, E_{16}, **E_{15}; E_{20}**, E_{23}, **E_{24}** (for E_4=**F**, E_5=**H**, E_{23}=**H**),
E_{14}, E_{10}, E_4, E_2, E_3, E_{22}, **E_{24}; E_{15}**, E_{13}, E_8, E_6, E_{12}, E_{21}, **E_{20}** (for E_4=**H**, E_5=**F**, E_{23}=**F**),
E_{14}, E_{10}, E_9, E_{13}, **E_{15}**; **E_{20}**, E_{21}**,** E_{12}, E_6, E_5, E_2, E_3, E_{22}, **E_{24}** (for E_4=**F**, E_5=**H**, E_{23}=**F**).

For $E_{24} + E_{20} + E_{15} + E_{14} = 2$**H** + 2**F,** B-set values are presented by Table 5.

Table 5: B-set values for the Fig.4 example

	$B_1=E_{24}$	$B_2=E_{20}$	$B_3=E_{15}$	$B_4=E_{14}$
1	**H**	**H**	**F**	**F**
2	**H**	**F**	**H**	**F**
3	**H**	**F**	**F**	**H**
4	**F**	**H**	**H**	**F**
5	**F**	**H**	**F**	**H**
6	**F**	**F**	**H**	**H**

For these, there are five feasible pp-solutions:

1. E_{20}, E_{21}, E_{16}, E_{13}, E_9, E_{10}, E_1**,** E_2, E_5, E_6, E_7, E_{22}, **E_{24}** (for E_4=**F**, E_5=**H**, E_{23}=**F**)
2. E_{15}, E_{13}, E_8, E_5, E_4, E_{10}, E_1**,** E_3, E_7, E_{12}, E_{21}, E_{23}, **E_{24}** (for E_4=**H**, E_5=**H**, E_{23}=**H**)
3. E_{14}, E_{10}, E_4, E_2, E_3, E_7, E_6**,** E_8, E_{13}, E_{16}, E_{21}, E_{23}, **E_{24}** (for E_4=**H**, E_5=**F**, E_{23}=**H**)
4. No solutions
5. E_{14}, E_{10}, E_9, E_{13}, E_{16}, E_{12}, E_6**,** E_5, E_2, E_3, E_{22}, E_{23}, **E_{20}** (for E_4=**F**, E_5=**H**, E_{23}=**H**)
6. E_{14}, E_{10}, E_4, E_2, E_3, E_{22}, E_{23}**,** E_{21}, E_{12}, E_6, E_8, E_{13}, **E_{15}** (for E_4=**H**, E_5=**F**, E_{23}=**H**).

Taking into account the redundancy elimination, we remove pp-solution

$\mathbf{E_{14}}, E_{10}, E_9, E_8, E_5, E_2, E_3\mathbf{,} E_7, E_{12}, E_{16}, \mathbf{E_{15};}\quad \mathbf{E_{20}}, E_{23}, \mathbf{E_{24}}$

and after that discover three **r**-invariants: $E_{13}=\mathbf{H}$**,** $E_2=E_6$, and $E_4=E_8$. The edges involved are not from the set of ($\mathbf{E_4}$, $\mathbf{E_5}$, $\mathbf{E_{23}}$); therefore the **r**-invariants are checked for independence from the **n**-invariants and previously extracted **r**-invariants. As a result of this check, we discover that $E_4=E_8$ is dependent (derived from the **n**-invariants and $E_{10}=\mathbf{H}$**,** $E_{13}=\mathbf{H}$)**.**

To continue extracting the invariants, we consider another region (see the right subgraph at Fig. 5).

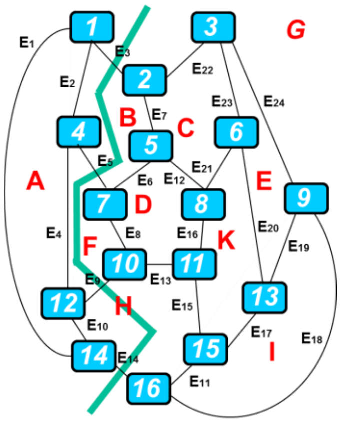


Fig. 5. A cut-set line based on the B-F-H-face path

In the system of the (2.3) type equations, we add the $E_{13}=\mathbf{H}$ invariant:

$$
\begin{array}{lll}
 & E_{22}+E_{23}+E_{24}=2\mathbf{H}+\mathbf{F} & E_{17}+E_{19}+E_{20}=2\mathbf{H}+\mathbf{F} \\
 & E_6+E_7+E_{12}=2\mathbf{H}+\mathbf{F} & E_{11}+E_{15}+E_{17}=2\mathbf{H}+\mathbf{F} \\
 & E_{20}+E_{21}+E_{23}=2\mathbf{H}+\mathbf{F} & E_7+E_{22}+\mathbf{E_3}=2\mathbf{H}+\mathbf{F} \quad (3.2) \\
 & E_{12}+E_{16}+E_{21}=2\mathbf{H}+\mathbf{F} & E_6+E_8+\mathbf{E_5}=2\mathbf{H}+\mathbf{F} \\
 & E_{18}+E_{19}+E_{24}=2\mathbf{H}+\mathbf{F} & E_8+E_{13}+\mathbf{E_9}=2\mathbf{H}+\mathbf{F} \\
E_{13}=\mathbf{H} & E_{13}+E_{15}+E_{16}=2\mathbf{H}+\mathbf{F} & E_{11}+E_{18}+\mathbf{E_{14}}=2\mathbf{H}+\mathbf{F}.
\end{array}
$$

The B-set values are sets of values of $\mathbf{E_3}$, $\mathbf{E_5}$, $\mathbf{E_9}$, and $\mathbf{E_{14}}$. Taking names of these edges as symbolic parameters and solving system (3.2) (where 13 equations and 16 unknowns), we find three independent edges (E_{19}, E_{23} and E_{24}). By considering possible value variations of these independent edges, as well as B-edges, we find all possible pp-solutions in this new region. As for the previous region, the B-edge value has to satisfy one of the equalities: $E_3+E_5+E_9+E_{14}=4\mathbf{H}$ or $E_3+E_5+E_9+E_{14}=2\mathbf{H}+2\mathbf{F}$**.**

For $E_3+E_5+E_9+E_{14}=4\mathbf{H}$ there are four valid pp-solutions:

$\mathbf{E_{14}}, E_{11}, E_{17}, E_{19}, E_{24}, E_{23}, E_{21}\mathbf{,} E_{16}, E_{13}, \mathbf{E_9;}\ \ \mathbf{E_5}, E_6, E_7, \mathbf{E_3}$ (for $E_{19}=\mathbf{H}$, $E_{23}=\mathbf{H}$, $E_{24}=\mathbf{H}$),
$\mathbf{E_{14}}, E_{18}, E_9, E_{17}, E_{15}, E_{13}, \mathbf{E_9;}\ \mathbf{E_5}, E_6, E_{12}, E_{21}, E_{23}, E_{22}, \mathbf{E_3}$ (for $E_{19}=\mathbf{H}$, $E_{23}=\mathbf{H}$,$E_{24}=\mathbf{F}$),

E_{14}, E_{11}, E_{15}, E_{13}, **E_{9}**; **E_{5}**, E_{6}, E_{12}, E_{21}, E_{20}, E_{19}, E_{24}, E_{22}, **E_{3}** (for E_{19}=**H**, E_{23}=**F**, E_{24}=**H**),
E_{14}, E_{18}, E_{24}, E_{22}, **E_{3}**; **E_{5}**, E_{6}, E_{12}, E_{21}, E_{20}, E_{17}, E_{15}, E_{13}, **E_{9}** (for E_{19}=**F**, E_{23}=**F**, E_{24}=**H**).

For $E_3 + E_5 + E_9 + E_{14}$ = 2**H** + 2**F** there are five feasible pp-solutions:
At B-set value (E_3=**H**, E_5=**F**, E_9=**F**, E_{14}=**H**)
E_{14}, E_{11}, E_{17}, E_{19}, E_{24}, E_{23}, E_{21}, E_{16}, E_{13}, E_{8}, E_{6}, E_{7}, **E_{3}** (for E_{19}=**H**, E_{23}=**H**, E_{24}=**H**),
E_{14}, E_{18}, E_{19}, E_{17}, E_{15}, E_{13}, E_{8}, E_{6}, E_{12}, E_{21}, E_{23}, E_{22}, **E_{3}** (for E_{19}=**H**, E_{23}=**H**, E_{24}=**F**),
E_{14}, E_{11}, E_{15}, E_{13}, E_{8}, E_{6}, E_{12}, E_{21}, E_{20}, E_{19}, E_{24}, E_{22}, **E_{3}** (for E_{19}=**H**, E_{23}=**F**, E_{24}=**H**)

At B-set value (E_3=**F**, E_5=**H**, E_9=**H**, E_{14}=**F**)
E_{5}, E_{6}, E_{7}, E_{22}, E_{24}, E_{18}, E_{11}, E_{17}, E_{20}, E_{21}, E_{16}, E_{13}, **E_{9}** (for E_{19}=**F**, E_{23}=**F**, E_{24}=**H**).

At B-set value (E_3=**F**, E_5=**H**, E_9=**F**, E_{14}=**H**)
E_{14}, E_{18}, E_{24}, E_{22}, E_{7}, E_{12}, E_{21}, E_{20}, E_{17}, E_{15}, E_{13}, E_{8}, **E_{5}** (for E_{19}=**F**, E_{23}=**F**, E_{24}=**H**).

From these pp-solutions, we extract ind-set values (presented in Table 6). By preserving column 3 from cluster 1-3 and column 7 from cluster 5-7, we extract two invariants: E_{24}=**H** and E_{23}=**F** (in cases of reasoning on symbolic expressions of the edge values, instead of invariants of the E_{23}=**F** type, it is possible to use a pair of the <E_{22}=**H**, E_{21}=**H**> type).

Table 6: Independent edge values

	1	2	3	4	5	6	7	8	9
E_{19}	**H**	**H**	**H**	**F**	**H**	**H**	**H**	**F**	**F**
E_{23}	**H**	**H**	**F**	**F**	**H**	**H**	**F**	**F**	**F**
E_{24}	**H**	**F**	**H**	**H**	**H**	**F**	**H**	**H**	**H**

To continue extracting the invariants, we consider a new region (see the bottom subgraph at Fig. 6).

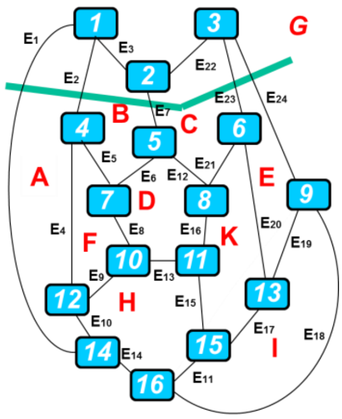


Fig. 6. A cut-set line based on the A-B-C-E-face path

In the system of the (2.3) type equations, we add invariants extracted before and applicable to this region. In addition, we analyze their influence on B-set values. As a result, the B-set is reduced to two edges (E_1 and E_7; E_2 is defined by them) and ind-set is reduced to edges E_4 and E_{11}. For various values of the B-edges, there are five pp-solutions:

At B-set value (E_7=**H**, E_1=**H**)
E_1, E_{10}, E_4, E_2; E_7, E_6, E_8**,** E_{13}, E_{16}, E_{21}, E_{20}, E_{17}, E_{11}, E_{18}, **E_{24}** (for E_4=**H**, E_{11}=**H**),
E_1, E_{10}, E_9, E_{13}, E_{16}, E_{21}, E_{20}**,** E_{17}, E_{11}, E_{18}, E_{24}; E_7, E_6, E_5, **E_2** (for E_4=**F**, E_{11}=**H**).

At B-set value (E_7=**H**, E_1=**F**)
E_7, E_{12}, E_{21}, E_{20}, E_{17}, E_{15}, E_{13}**,** E_8, E_5, E_4, E_{10}, E_{14}, E_{18}, **E_{24}** (for E_4=**H**, E_{11}=**F**).

At B-set value (E_7=**F**, E_1=**F**)
E_2, E_4, E_{10}, E_{14}, E_{11}, E_{15}, E_{13}**,** E_8, E_6, E_{12}, E_{21}, E_{20}, E_{19}, **E_{24}** (for E_4=**H**, E_{11}=**H**),
E_2, E_5, E_6, E_{12}, E_{21}, E_{20}, E_{17}**,** E_{15}, E_{13}, E_9, E_{10}, E_{14}, E_{18}, **E_{24}** (for E_4=**F**, E_{11}=**F**).

From these pp-solutions, $E_4+E_5+E_{11}=2\mathbf{H}+\mathbf{F}$ is extracted as the invariant.

In a similar way, we can consider additional regions and obtain additional invariants. However, in our case we have already got six invariants; therefore, we can consider the system of equations for the graph as a whole.

$$
\begin{array}{ll}
E_1 + E_2 + E_3 = 2\mathbf{H} + \mathbf{F} & E_3 + E_7 + E_{22} = 2\mathbf{H} + \mathbf{F} \\
E_2 + E_4 + E_5 = 2\mathbf{H} + \mathbf{F} & E_{12} + E_{16} + E_{21} = 2\mathbf{H} + \mathbf{F} \\
E_4 + E_9 + E_{10} = 2\mathbf{H} + \mathbf{F} & E_{22} + E_{23} + E_{24} = 2\mathbf{H} + \mathbf{F} \\
E_5 + E_6 + E_8 = 2\mathbf{H} + \mathbf{F} & E_{20} + E_{21} + E_{23} = 2\mathbf{H} + \mathbf{F} \\
E_6 + E_7 + E_{12} = 2\mathbf{H} + \mathbf{F} & E_{13} + E_{15} + E_{16} = 2\mathbf{H} + \mathbf{F} \\
E_8 + E_9 + E_{13} = 2\mathbf{H} + \mathbf{F} & E_1 + E_{10} + E_{14} = 2\mathbf{H} + \mathbf{F} \\
E_{17} + E_{19} + E_{20} = 2\mathbf{H} + \mathbf{F} & E_{11} + E_{15} + E_{17} = 2\mathbf{H} + \mathbf{F} \\
E_{18} + E_{19} + E_{24} = 2\mathbf{H} + \mathbf{F} & E_{11} + E_{14} + E_{18} = 2\mathbf{H} + \mathbf{F} \\
E_{10} = \mathbf{H} & E_{13} = \mathbf{H} \\
E_{24} = \mathbf{H} & E_{22} = \mathbf{H},\ E_{20} = \mathbf{H} \text{ (instead of } E_{23} = \mathbf{F}) \\
E_2 = E_6 & E_4 + E_5 + E_{11} = 2\mathbf{H} + \mathbf{F}.
\end{array}
\qquad (3.3)
$$

The solver provides two independent edges E_1 and E_4 to get unique solutions for (3.3). There are no solutions for E_1=E_4, and there are **two Hamiltonian cycles** for $E_1 \neq E_4$:
{E_1, E_{10}, E_9, E_{13}, E_{16}, E_{21}, E_{20}**,** E_{17}, E_{11}, E_{18}, E_{24}, E_{22}, E_7, E_6, E_5, E_2} and
{E_{10}, E_{14}, E_{11}, E_{15}, E_{13}, E_8, E_6**,** E_{12}, E_{21}, E_{20}, E_{19}, E_{24}, E_{22}, E_3, E_2, E_4}, respectively for (E_1=**H**, E_4=**F**) and (E_1=**F**, E_4=**H**).

4. Redundancy related index for searching appropriate regions

A fundamental point of our approach is regional redundancy and "independence" represented by corresponding invariants. For searching the appropriate regions and their shapes, we consider around each graph node a starting region including, first of all, three faces incident to this node. Then, these regions are sorted according to the growth of absolute values of a redundancy related index (RR-index), that is defined as the number of interior faces in a region minus the number of the region B-edges.

The order mentioned is just used for some systematic search of the graph regions where RR-index ≥ 1, and where the first attempt of discovering **r**-invariants is performed.

For obtaining a region with RR-index ≥ 1, we analyze the neighborhood of a starting region and attach to it one or a few faces to arrange a new region with a larger RR-index. A number of such arrangement steps can be performed for getting a **basic region** with RR-index ≥ 1. To demonstrate this process, let's consider logarithmic size regions **A** and its neighbor **D** (Fig. 7), sharing some B-edges with **A**, and assume compound region **AD,** including **A** and **D,** is connected (otherwise another neighbor is selected). In addition, we assume that region **A** has V_A nodes and is surrounded by $K_{tlr} + K_b$ B-edges (K_{tlr} is the number of B-edges on top-left-right sides and K_b is the number of B-edges on the bottom side).

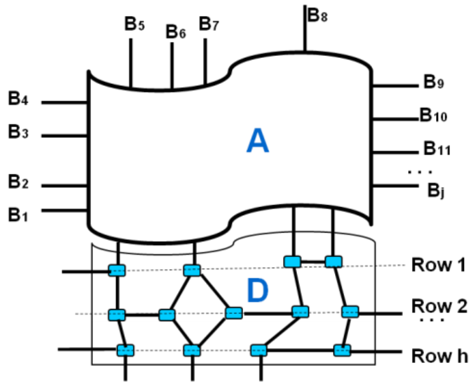


Fig. 7. Neighboring regions A and D

Applying the Euler formula we can obtain for interior faces of the region graph, the following expression:

$$F_A = V_A/2 – (K_{tlr} + K_b)/2 + 1.$$

It is a formula connecting the number of regional interior faces, the number of regional nodes, and the number of regional B-edges. Then, for RR-index (RRI), we can obtain $RRI_A = F_A – (K_{tlr} + K_b) = V_A/2 – 3(K_{tlr} + K_b)/2 + 1$.

Now, let region **D** have *h* internal rows representing nodes reachable by one step from nodes of the upper neighboring row: the nodes of the first row are reachable from the bottom nodes of region **A** by the bottom side B-edges, the nodes of the second row are reachable from the nodes of the first row by edges going down (from the first row nodes), and so on. Denote the number of nodes in row *j* by K_j, the average number of nodes (among all rows) by K and the number of B-edges on the bottom-left-right sides of region **D** by $\underline{K}_{blr}$. Then for the number of interior faces of compound region **AD** we can get:

$$F_{AD} = (V_A + h\cdot K)/2 - (K_{tlr} + \underline{K}_{blr})/2 + 1$$

and for the RR-index of compound region **AD**:

$$RRI_{AD} = F_{AD} - (K_{tlr} + \underline{K}_{blr}) = (V_A + h\cdot K)/2 - (K_{tlr} + \underline{K}_{blr})/2 + 1 - (K_{tlr} + \underline{K}_{blr}) \text{ or}$$

$$RRI_{AD} = RRI_A + h\cdot K/2 + 3(K_b - \underline{K}_{blr})/2. \qquad (4.1)$$

This formula shows how we can search regions with increasing values of RR-index. For a case with $\underline{K}_{blr} > K_b$ and $h\cdot K/2 + 3(K_b - \underline{K}_{blr})/2 \leq 0$, the shape of region **D** is modified by **increasing *h*** and/or **"tuning" rows** involved. For Fig. 7, tuning means removing the bottom-left node which is out of faces attached. In region **D** of Fig. 8 (a), there are 4 black nodes (labeled **b)** in the first row, 4 yellow and 4 white nodes in the second and third rows (labeled **y** and **w**, respectively), and five black nodes (also labeled **b)** in the fourth row;

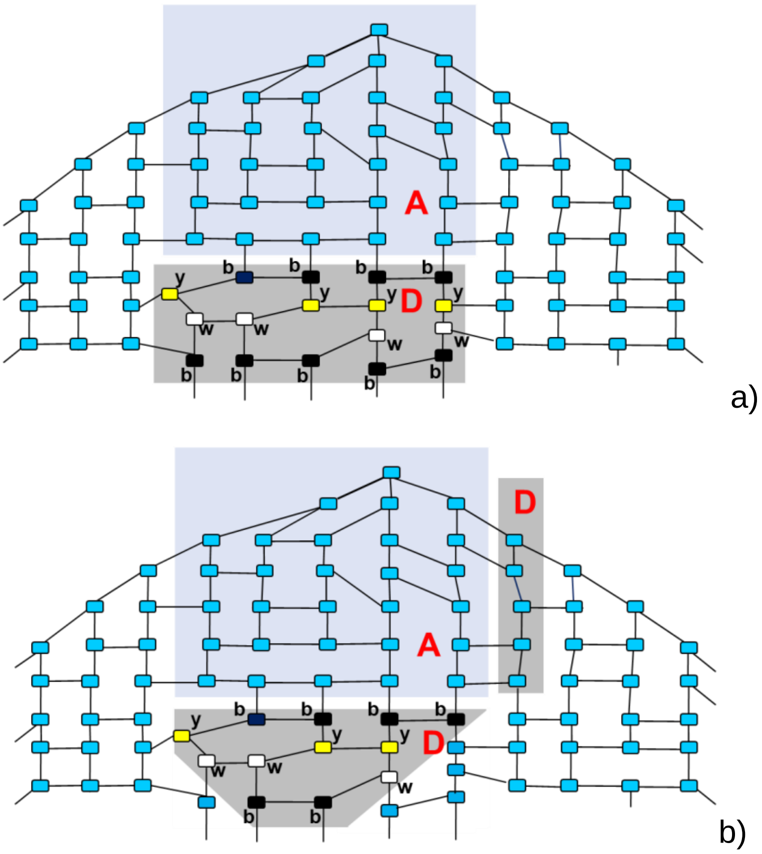


Fig. 8. Neighboring regions A and two D-s, one of which is with "tuning" rows

$RRI_A = -1$, $\underline{K}_{blr} > K_b$ but $h\cdot K/2 + 3(K_b - \underline{K}_{blr})/2 = 1$ and $RRI_{AD} = 0$.

However, "tuning" the shape of rows 2-4 decreases the value of $\underline{K}_{blr}$; see Fig. 8 (b), where there are 3 yellow nodes in the second row, 3 white nodes in the third rows, and two black nodes in the fourth row. As a result, $RRI_{AD'}$ = 2. In this example, such tuning operations are also applicable for two bottom rows of region **A** (before considering region **AD**) for removing two right nodes of these rows and for getting RRI_A = 1.

In general, tuning rows means decreasing a region cutting set by removing a face or by avoiding attachment of an incomplete face. In fact, the tuning operations are automatically performed, if **D** regions include only complete faces and their attachment increases the RR-index. A special point of attention is that different **D** regions lead to different increases; for example, attaching the right **D** at Fig. 8 (b) increases RRI_A by 4, while attaching the bottom **D** increases RRI_A by 3.

Formula (4.1) shows a "quadratic" growth of $h \cdot K/2$ and a "linear" growth of $3(K_b-\underline{K}_{blr})/2$. This means that basic regions with RR-index ≥ 1 are achievable.

The number of faces, in a sense, represents a "regional square" and the number of B-edges represents a "regional perimeter".

Every time, we need a basic region to start extracting r-invariants, the following operations are performed.

For a starting region **A,** we consider, one-after-another, possible locations of the region **D** attachment with h =1 and K_b = 2, 3,…,Ω (Ω is limited by a number of B-edges of region **A**). This is for discovering compound region **AD** with maximal RRI_{AD} (which has to be > RRI_A). Successful discovering leads us to assign this region **AD** for being a new region **A** and to restart the search. Unsuccessful discovering leads us to assign h = h+1 and after that to restart the search. It is finished by getting a basic region with RRI_{AD} = $\underline{©}$+1 (0≤$\underline{©}$≤©**,** © is a constant). These steps are a growing RR-index search.

We can consider our search from another point of view. For each starting region **A,** we try to attach a neighboring face (sharing some internal edges and B-edges with **A** and representing region **D**), which increases **RRI**. Absence of such a face suggests that we should try attaching a set of two or more neighboring faces (the number of these faces is limited and predefined). The failure of getting a result is used for selecting a set of faces preserving **RRI**. Then **AD** is assigned as a new **A,** and the search is restarted.

The basic region discovered is employed for extracting the **r**-invariants (this includes finding pp-solutions and applying redundancy eliminations) or (in the case of the invariant extracting failure) it is used for arranging neighboring regions (NRs) (see the next section and Appendix 3) and for applying the NR redundancy eliminations to unblock discovering invariants in the basic region. In a successful case, we use this basic region for discovering other **r**-invariants. After exhausting all possibilities in this region, we reuse one of the NRs (as a basic one) or go to a new starting region with the largest RR-index.

Discovering an **r**-invariant is followed by checking the number of the **r**-invariants available and finishing the search, if this number is ≥ ***N***/2 (it is also for checking the invariant independence and duplication); otherwise employing (if any) a non-considered-yet starting region. Results obtained in a region are saved and taken into account by further operations in this and other regions.

5. Searching invariants in basic and neighboring regions

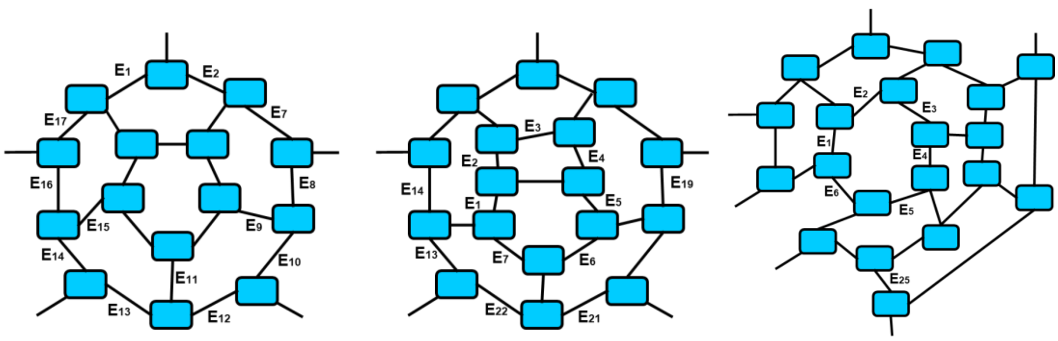
There are a lot of invariants in the graph regions. For example, let's select any graph node and consider all nodes reachable within two steps. Then, the sum of edges involved at the second step is always equal to 4**H**+2**F**. A similar invariant exists for edges involved at the third step and related to nodes reachable within three steps: the sum of these edges is equal to 8**H**+4**F**.

Another example is related to a region of one face with even degree *k*. Expression $E_1 - E_2 + E_3 - E_4 + \ldots + E_{k-1} - E_k$ (where edges neighboring in the expression are neighboring B-edges of the region) is always equal to 0. Unfortunately, they are derived from the **n**-invariants and cannot be used for the systems of equations mentioned above.

5.1. Examples of independent r-invariants

A great variety of basic regions allows directly discovering the independent **r**-invariants. For example, in the region of Fig. 9 (a) with RRI=1, five **r**-invariants (E_1=**H**, E_7=**H**, E_{10}=**H**, E_{13}=**H** and E_{16}=**H**) or alternatively (E_2=**H**, E_8=**H**, E_{12}=**H**, E_{14}=**H** and E_{17}=**H**) can be extracted. In the region of Fig.9 (b) with RRI=2, **r**-invariants $E_1=E_4$ and $E_2=E_5$ are simultaneously available (however, they depend on each other); alternatively, $E_{19}=E_{21}$ or $E_{14}=E_{22}$ can also be discovered. In the region of Fig. 9 (c) with RRI=1, E_{25}=**H** is the invariant.

R-invariants also exist in regions presented by Fig.10 (a-d). $E_2=E_5$ (or $E_3=E_6$), $E_1=E_3$ (or $E_9=E_{11}$), E_{12}=**H** and $E_1=E_8$ (or $E_1=E_6$) can respectively be discovered in Fig.10 (a) - (d). Even for cases with RRI = 0 and RRI = -1, **r**-invariants can



a) b) c)

Fig. 9. Examples of regions with r-invariants

exist in regions with four B-edges.

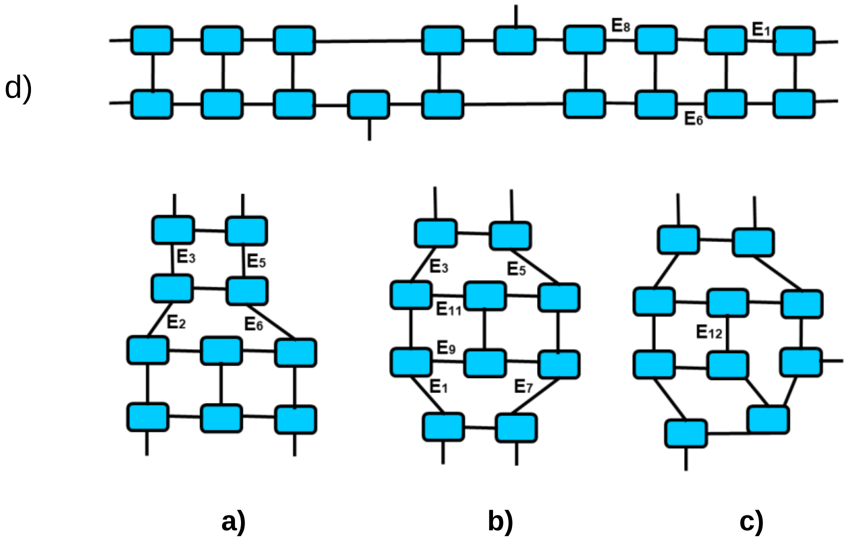


Fig. 10. Additional examples of regions with r-invariants

It is important to note that *an r-invariant extracted from a set of pp-solutions in a region is valid on any subset of these pp-solutions. This means that additional removal of pp-solutions from the set preserves the r-invariant extracted.*

However, there are basic regions without invariants we are interested in; Fig.11 presents an example of such a region. Therefore (within the framework of the polynomial complexity), we consider some neighboring regions, overlapping the region under consideration (RUC), their pp-solutions and appropriate redundancy eliminations for decreasing the number of pp-solutions in RUC and, as a result, for discovering the **r**-invariants there.

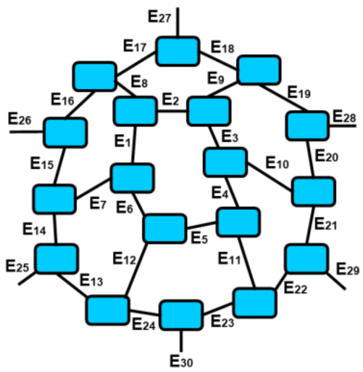


Fig. 11. A region without invariants

5.2. Arranging the neighboring regions

There can be a variety of approaches. Ours is based on selecting an invariant pattern (a candidate to be an **r**-invariant) for a basic RUC and discovering pp-solutions that block the recognition of this pattern as the **r**-invariant (E_j=**H,** E_j+E_k=**H**+**F** and $E_i+E_j+E_k$=2**H**+**F** are examples of patterns to be considered). For the blocking pp-solutions, we select (E_j,E_k) (and/or (B_i,B_p) and (E_i,B_p), where B_i and B_p are B-edges) pairs of the RUC edges satisfying the equality of $E_j=E_k$=**F.** Such pairs are used for orientation within arranging an NR. The orientation means that a part of the pairs should be inside this NR.

Table 7 shows 15 clusters of pp-solutions related to the region presented by Fig.11. Each cluster includes two pp-solutions and any of these two can be eliminated without altering the graph Hamiltonicity. The total number of the clusters is 29, however we selected 15; they possess all features we are interested in to present. Though we cannot immediately discover an **r**-invariant, we can select an invariant pattern (for example, E_{14}=**H**) and analyze conditions blocking the recognition of this pattern as the **r**-invariant. Within the cluster eliminations, pattern E_{14} = **H** can be recognized as an **r**-invariant in all

Table 7: A set of pp-solutions for the region presented by Fig. 11

E_1	F,F	H,H	H,H	F,F	H,H	H,H	H,H	H,H	H,H	H,F	H,F	H,H	H,F	H,H	H,H
E_2	H,H	F,H	H,F	H,H	H,H	H,H	H,F	F,F	H,H	F,H	F,H	H,F	F,H	H,H	H,H
E_3	H,H	H,F	F,H	H,F	H,F	H,H	F,H	H,H	H,F	H,H	H,H	F,H	H,F	F,H	F,F
E_4	H,F	H,H	H,H	F,H	F,H	H,F	H,F	F,H	F,H	H,F	F,F	H,H	F,H	H,F	H,H
E_5	F,H	F,H	F,F	H,H	H,H	F,H	F,H	H,F	H,H	F,H	H,H	H,H	H,H	F,H	F,H
E_6	H,H	H,F	H,H	H,H	H,H	H,H	H,H	H,H	F,F	H,H	H,H	F,F	H,H	H,F	H,F
E_7	H,H	F,H	F,F	H,H	F,F	F,F	F,F	F,F	H,H	F,H	F,H	H,H	F,H	F,H	F,H
E_8	H,H	H,F	F,H	H,H	F,F	F,F	F,H	H,H	F,F	H,H	H,H	F,H	H,H	F,F	F,F
E_9	F,F	H,H	H,H	F,H	F,H	F,F	H,H	H,H	F,H	H,F	H,F	H,H	H,H	H,F	H,H
E_{10}	F,H	F,H	H,F	H,H	H,H	F,H	H,H	H,F	H,H	F,H	H,H	H,F	H,H	H,H	H,H
E_{11}	H,H	H,F	H,H	H,F	H,F	H,H	H,H	H,H	H,F	H,H	H,H	F,F	H,F	H,H	H,F
E_{12}	H,F	H,H	H,H	F,F	F,F	H,F	H,F	F,H	H,H	H,F	F,F	H,H	F,F	H,H	H,H
E_{13}	H,H	F,H	F,F	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	F,H	F,H
E_{14}	F,F	H,F	H,H	F,F	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,F	H,F
E_{15}	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	F,F	H,F	H,F	F,F	H,F	H,H	H,H
E_{16}	H,H	H,H	H,H	H,H	H,H	H,H	H,H	F,F	H,H	F,H	F,H	H,H	F,H	H,H	H,H
E_{17}	F,F	F,H	H,F	F,F	H,H	H,H	H,F	H,H	H,H	H,F	H,F	H,F	H,F	H,H	H,H
E_{18}	H,H	H,F	F,H	H,H	H,H	H,H	F,H	H,H	H,H	F,H	F,H	F,H	F,H	H,H	F,F
E_{19}	H,H	F,H	H,F	H,F	H,F	H,H	H,F	F,F	H,F	H,H	H,H	H,F	H,F	F,H	H,H
E_{20}	H,H	H,F	F,H	F,H	F,H	H,H	F,H	H,H	F,H	H,H	F,F	F,H	F,H	H,F	H,H
E_{21}	H,F	H,H	H,H	H,F	H,F	H,F	H,F	F,H	H,F	H,F	H,H	H,H	H,F	F,H	F,F
E_{22}	F,H	H,H	F,F	F,H	F,H	F,H	F,H	H,F	F,H	F,H	H,H	H,H	F,H	H,F	H,H
E_{23}	H,F	F,H	H,H	H,H	H,H	H,F	H,F	F,H	H,H	H,F	F,F	H,H	H,H	F,H	F,H
E_{24}	F,H	H,F	H,H	H,H	H,H	F,H	F,H	H,F	F,F	F,H	H,H	F,F	H,H	H,F	H,F
E_{25}	H,H	H,H	H,H	H,H	F,F	F,F	F,F	F,F	F,F	F,F	F,F	F,F	F,F	H,H	H,H
E_{26}	F,F	F,F	F,F	F,F	F,F	F,F	F,F	H,H	H,H	H,H	H,H	H,H	H,H	F,F	F,F
E_{27}	H,H	H,H	H,H	H,H	F,F	F,F	H,H	F,F	F,F	H,H	H,H	H,H	H,H	F,F	H,H
E_{28}	F,F	H,H	H,H	H,H	H,H	F,F	H,H	H,H	H,H	F,F	H,H	H,H	H,H	H,H	F,F
E_{29}	H,H	F,F	H,H	H,H	H,H	H,H	H,H	H,H	H,H	H,H	F,F	F,F	H,H	H,H	H,H
E_{30}	H,H	H,H	F,F	F,F	F,F	H,H	H,H	H,H	H,H	H,H	H,H	H,H	F,F	H,H	H,H

clusters (including those we omitted for showing) except two of them (grey-lighted by a grey background of characters); all pp-solutions in these two clusters have E_{14} = **F**.

In addition, in the left grey-lighted cluster all (in our case, just two) pp-solutions have E_1=E_9=E_{17}=B_{26}=B_{28}=**F** and in the right grey-lighted cluster all pp-solutions have E_1=E_{12}=E_{17}=B_{26}=B_{30}=**F**. In order to recognize the pattern as an **r**-invariant, in some way, pp-solutions with E_{14} = **F** should be eliminated in the RUC. Such effect eliminations can be reached in an NR which overlaps the RUC and includes the edges mentioned above as internal ones.

Redundancy eliminations in such an NR removing pp-solutions with, for example, B_{26}=B_{28}=**F** and E_1=E_{17}=**F,** provides the necessary result; we call such eliminations 2**F** pair eliminations. In fact, the result could be provided by any pair from (E_1, E_9, E_{17}, B_{26}, B_{28}) instead of B_{26}=B_{28}=**F** and any pair from (E_1, E_{12}, E_{17}, B_{26}, B_{30}) instead of B_1=B_{17}=**F**.

Considering Fig.11 as an example of the NR region and applying some redundancy elimination scheme, we can obtain that in each of the following pairs, edges are never simultaneously equal to **F**:

(E_1,E_4), (E_1,E_{15}), (E_2,E_{17}), (E_2,E_{21}), (E_4,E_{21}), (E_5,E_{15}), (E_5,E_{23}), (E_{15},E_{17}), (E_{17},E_{23}), (E_{21},E_{23}).

In addition, by "rotating" the elimination scheme, we can get six different sets of such pair types;   among them is

(E_1,E_{16}), (E_1,E_{20}), (E_3,E_6), (E_3,E_{20}), (E_4,E_{14}), (E_4,E_{22}), (E_6,E_{14}), (E_{14},E_{16}), (E_{16},E_{22}), (E_{20},E_{22}).

For arranging an NR (see Fig. 12 (a)-(d)), we select a part of its B-edges inside RUC and extend the overlapping subregion into a basic region (as a rule, with RRI ≥ 1) possessing pairs of edges that can unblock the invariant recognition in the RUC. Fig.12 (a) depicts the Fig.11 region as a basic one and an NR including E_1, E_3, E_{15} and E_{20} as its B-edges and B_{26} and B_{28} as its internal edges. The redundancy eliminations, we are trying to find in this NR, are related (the edges involved are blue) to E_{17}=B_{26}=**F** (for the right grey-lighted cluster) and one of the following cases: E_9=E_{17}=**F**, E_9=B_{26}=**F**, E_9=B_{28}=**F,** E_{17}=B_{26}=**F,** E_{17}=B_{28}=**F** and B_{26}=B_{28}=**F** (for the left grey-lighted cluster).

Fig.12 (b) also depicts the Fig.11 region as a basic one and the NR including E_7, E_{11}, E_{12}, E_{15} and E_{21} as its B-edges and B_{26} and B_{28} as its internal edges. The NR redundancy eliminations involved are related to any two pairs: one of which is taken from E_1=E_{17}=B_{26}=**F** and another from E_1=E_9=E_{17}=B_{26}=B_{28}=**F**.

Fig.12 (c) is similar to Fig.12 (b). The difference is in the RUC pp-solution elimination applied in advance for the right grey-lighted cluster (of Table 7). As a result, the redundancy eliminations (REs), we are trying to find in the NR, are related to any two pairs: one of which is taken from E_1=E_4=E_9=E_{17}=

E20=B26=**F** (the green elements of the pp-solution preserved in the right grey-lighted cluster) and another from E1=E9=E17=B26=B28=**F** (the green elements of pp-solutions in the left grey-lighted cluster). In all cases related to Fig.12 (a-c), the pair sets are overlapped; therefore, some single pairs are attempted for both clusters.

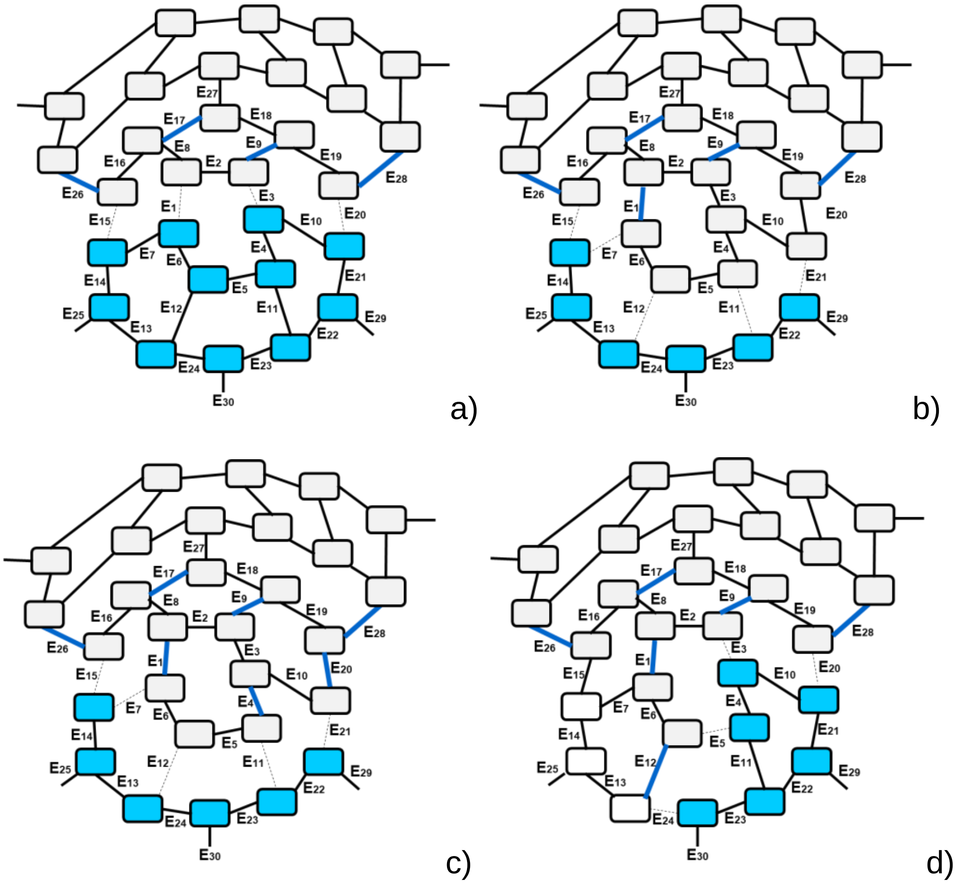


Fig.12. Examples of arranging NRs

In the examples provided, edges involved in the invariant pattern have been excluded from the NRs. However, the arrangement of NRs with such edges located internally is also possible. Fig.12 (d) depicts such a case, where the REs we are trying to find are related to any two pairs: one of which is taken from E1=E9=E12=E17=B26=F and another from E1=E9=E17=B26=B28=F (such an NR can be considered as another basic region).

The first step of arranging the NR, as a rule, includes considering a few directions (of “top, bottom, left and right” types) from RUC for getting a joint elimination impact of a few NRs; in Fig. 13 (a), these directions are presented by NR(t), NR(b), NR(l) and NR(r). In addition, to enhance this impact, alternative eliminations are checked for each direction to decrease the number of blocking pp-solutions in the RUC and in a corresponding NR. The joint elimination failure leads (for the second step) to searching all directions from an NR with minimum number of blocking pp-solutions. In Fig. 13 (b),

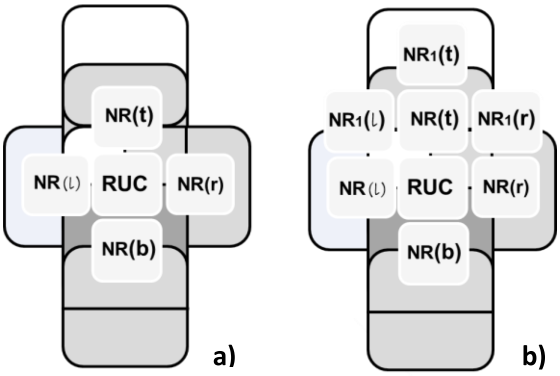


Fig.13. Directions of arranging NRs

NR(t) is presented as such an NR and NR_1(t), NR_1(l) and NR_1(r) as its directions for getting a joint elimination impact. The new joint elimination failure leads, in its turn, to considering the directions from a new NR which is selected among all "boundary" NRs (in Fig.13 (b), they are NR_1(t), NR_1(l), NR_1(r), NR(b), NR(l) and NR(r)). In such a way (like a minimum spanning tree), NRs covering the graph and their tree-type chains are organized. Other details of the NR arrangement within NR chains are presented in the next subsection.

5.3. 2F pair eliminations in the neighboring regions

2**F** pair eliminations are based on Hamiltonicity-preserving REs applying in an NR or in a remote NR (with impact through a chain of NRs) for removing pp-solutions blocking the invariant recognition in the basic region. Eliminating the NR pp-solutions is related to the edge pair values of the $B_j=B_k=$**F** **(**and/or $E_i=E_p=$**F**, $E_i=B_p=$**F**) type.

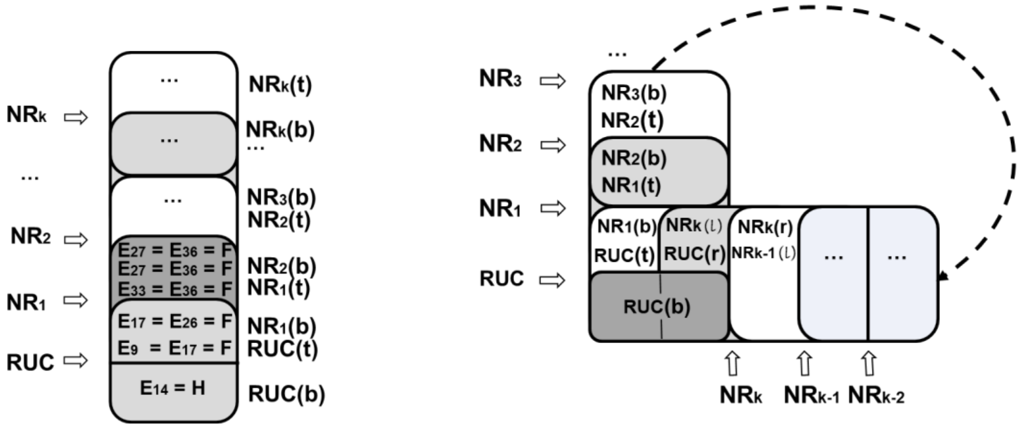
- Successful REs in the NR unblock pp-solutions in the RUC and allow discovering the RUC invariants. In order to reach this goal, we check REs for each edge pair (above mentioned type) related to each blocking pp-solution in the RUC (pairs are from an overlapping area of the RUC and NR). The success means that in the remaining NR pp-solutions, there are not pair edges (in the area pointed to) simultaneously equal to **F**. Such success for one pair is enough for removing a related blocking pp-solution. In fact, REs for each direction from RUC and their joint impact are checked.
- Unsuccessful REs (or partially successful REs) lead to selecting an NR with a minimum number of blocking pp-solutions and to extracting from this NR pp-solutions which block the success at the current step; these pp-solutions are based on edge pairs from the NR area which is not overlapped with the RUC. These pairs are applied for removing the blocking NR pp-solutions by REs in a new NR (the new NR direction is selected on rules mentioned in the previous subsection). Successful REs

(including the joint REs) in this new NR leads to unblocking pp-solutions in a chain of the previous NRs and discovering the invariant in the RUC.

A blocking NR pp-solution is valid, if it is not involved in internal circles of a compound region including the RUC and NRs (see Appendix 4).

The unblocking failure leads to a new-new NR and so on. In such a way, a chain (a tree branch) of NRs is created and extended. In each NR of the chain, a set of blocking pp-solutions is discovered (removing these pp-solutions unblocks blocking pp-solutions in the previous NRs and in the RUC). The chain is extended until the unfavorable conditions are discovered in the newest NR, then another chain is activated (the unfavorable conditions are the large number of blocking pp-solutions or absence of the graph space (including fully covering the graph)).

- Fig.14 illustrates 2**F** pair elimination issues related to the NR chain growth (at each step we show one direction related to an NR with the minimum number of blocking pp-solutions). At Fig.14 (a) a RUC is a point for starting an NR chain. An invariant pattern (let it be E_{14} = **H)** is based on the edge from RUC(b) (the bottom part of the RUC). Edge pairs that can eliminate the RUC pp-solutions with E_{14} = **F** are in RUC(t) (the top part of the RUC), as well as in the b-part of the first NR (NR_1(b)) overlapped with RUC(t); let them be E_{17}=B_{26}=**F** and E_9=E_{17}=**F** (E_{17}=B_{26}=**F** is related to one part of the RUC pp-solutions with E_{14} = **F**, and E_9=E_{17}=**F** is related to another part; the parts can be overlapped).



a) b)

Fig. 14. Examples of the NR chains

- The failure to eliminate all pp-solutions in NR_1 with simultaneous values of E_{17}=B_{26}=**F** and E_9=E_{17}=**F** leads to discovering the NR_1(t) edge pairs that can be applied for possible eliminations of pp-solutions in NR_2; let them be E_{27}=B_{36}=**F**, E_{27}=B_{38}=**F** and E_{33}=E_{36}=**F**. Then we attempt to eliminate all pp-solutions in NR_2 with simultaneous values of E_{27}=B_{36}=**F**, E_{27}=B_{38}=**F** and E_{33}=E_{36}=**F**. The failure leads to discovering the NR_2(t) edge pairs that can be applied for possible eliminations of pp-solutions in NR_3. In this way, we can go through the whole graph. In Fig.14 (b) the NR chain makes a loop by returning to the RUC (in fact, the return can be to any previous NR).

- The failure to discover the unblocking pp-solutions in NR_{k-1} requires finding the 2**F** pairs in the left part of NR_{k-1} ($NR_{k-1}(l)$ which is overlapped with $NR_k(r)$ (the right part of NR_k)). Then, the absence of blocking pp-solutions is searched for in NR_k.

The successful search means that among these pp-solutions there are no 2**F** blocking pairs from $NR_{k-1}(l)$ and corresponding REs unblock pp-solutions in NR_{k-1},…, NR_2, NR_1. Such a remote impact allows recognizing the invariant pattern in the RUC.

The unsuccessful search leads to a deadlock finding the 2**F** pairs in the RUC pp-solutions. This means that removing pp-solutions depends on removing themselves. Then, we cancel the return to the RUC and take another direction (if the graph space is available) or activate another chain.

In such a way, we continue arranging the NR chains (based on directions of an NR with minimum number of blocking pp-solutions) for covering a larger and larger part of ***G*** and discovering the unblocking pp-solutions.

However, we can assume a special aspect of the search. It is related to including all NR pp-solutions in blocking pp-solutions. Such a situation can be assumable because of REs and some other operations (see the next subsection) used for recognizing previous invariant patterns.

All NR pp-solutions in blocking pp-solutions mean that the current invariant pattern cannot be recognized.

Then, we return to the RUC for selecting a complementing invariant pattern (for example, for E_{14} = **H** it can be E_{13} = **H**).

Complementing patterns cover all possible pp-solutions (as well as, global solutions); failures to recognize all complementing patterns mean the absence of Hamiltonicity.

In the next subsection, we continue the consideration of covering graph ***G*** by a tree-type composition of the region chains for recognizing an invariant pattern in the root region of this composition. For this goal, Hamiltonicity-preserving operations of the regional redundancy and the redundancy with a remote impact are employed. They allow obtaining pp-solutions in all regions and recognizing the invariant pattern or discovering a Hamiltonian cycle. For the smooth composition of the tree-type chains, a flexible region arrangement based on including additional graph space into a region or on redesigning the regional neighborhood is naturally expected.

5.4. Invariant pattern recognition in tree-type chains

As we have already mentioned, the minimum number of blocking pp-solutions is applied as a criterion for arranging the tree-type chains of NRs covering the graph. At each step of this arranging, all "boundary" NRs are analyzed for selecting an NR which is attached to a corresponding chain. The arranging goal is in discovering the confirmation of 2**F** pair eliminations related to blocking pp-solutions of the RUC. A partial or full success unblocks corresponding RUC pp-solutions remotely through the chain.

In such a way, we arrange and scan all regions covering graph ***G***, and within this process can get success presented by necessary unblocking recognitions of 2**F** pair eliminations and applying them. On the other hand, after arranging and scanning regions covering graph ***G,*** we can get failure presented by existence of blocking pp-solutions in leaf regions of the tree chains. This means that the invariant pattern cannot be recognized because of potential existence of global solutions including the blocking pp-solutions in the RUC. The only chance is to show that all these global solutions are invalid.

In order to reach this goal, we attempt to solve the Hamiltonicity problem on graph ***G***, but at employing in the regions only the blocking pp-solutions. First, we require the involvement of all RUC's pp-solutions infeasible for the invariant pattern (because the presence of these pp-solutions blocks the invariant pattern recognition). Then, in each NR, we also take into account all blocking pp-solutions and exclude other pp-solutions.

Involving a pp-solution, which is not a blocking one in an NR, means an assumption of a global solution skipping all blocking pp-solutions in this NR; but this skipping leads to unblocking the blocking pp-solutions in a chain of NRs and in the RUC.

After that, based only on blocking pp-solutions, in the RUC and each NR, a necessary number of **r**-invariants are extracted. Then, a system of the equations is composed and solved. Existence of a solution gives us a Hamiltonian cycle; absence of a solution recognizes the invariant pattern in the RUC. The number of the blocking pp-solutions in a region is less that the number of all pp-solutions. This is favorable for extracting **r**-invariants from the blocking pp-solutions. Nevertheless, we can assume that there are not enough **r**-invariants for composing the system of equations. Then, we select an NR, as a new basic region (a new RUC), and an invariant pattern in it (in fact, the old RUC and a new invariant pattern in it can be applied for exhausting the RUC possibilities). After that, we repeat all operations presented above for recognizing the invariant pattern in the new RUC (and invariant patterns in NRs) and solving the Hamiltonicity problem related to the new slice of blocking pp-solutions.

It is important to note, that there are two fundamental points in our approach. The first is related to the polynomial complexity. We pay special attention to it in the next subsection. The second is related to lossless solution operations, which are based on Hamiltonicity-preserving eliminations and on separations of blocking pp-solutions from pp-solutions recognizing the invariant patterns. The separation operations (as well as the elimination ones) can apply alternative choices for arranging the blocking pp-solution slices across the regions. However, any choice is sufficient for recognizing the invariant pattern (in a case of non-Hamiltonicity of the graph where only the blocking pp-solutions in the regions are used) or for getting the problem solution (in a case of Hamiltonicity of the graph just mentioned).

5.5. Polynomial complexity of the approach

In order to demonstrate the polynomial complexity of the approach, we present it in two parts. In the first part, we create a binary tree with a polynomial number of nodes representing Hamiltonicity subproblems. Different nodes are represented by different slices of pp-solutions employed across all regions of graph ***G.*** The "thickness" of the leaf node slices allows discovering the necessary number of **r**-invariants. This discovering is performed in the second part, which also includes composing the systems of equations for the leaf nodes and solving the systems. In addition, a final result, based on results in the nodes, is obtained. Splitting the problem into a polynomial number of subproblems depends on possible schemes related to recognizing the invariant patterns. Fig. 14 -16 depicts four basic schemes.

Scheme 1 (Fig. 15) assumes that blocking pp-solutions, extracted at each step, and pp-solutions recognizing the invariant pattern, are split into two (more or less equal) parts in each NR. Then, within the first part, the splitting step is based on selecting a new invariant pattern (in one of the regions) and discovering subsets of corresponding blocking and recognizing pp-solutions. In such a case, after $log_2 N$ steps the number of pp-solutions in each NR is decreased by factor ***N***. The number of leaf nodes in the scheme tree is of the $2^{log_2 N} = N$ level (if necessary, we can take even more steps; for example, $log_2 N + n$ steps with $n < log_2 N$; in any case, the number of pp-solutions in an NR is much lesser than N^2). As a result, within the second part, it is discovered that the number of pp-solutions in each region of each leaf node is decreased to allow extracting the necessary invariants and solving the corresponding system of equations. A positive solution, at least in one leaf node, gives us a Hamiltonian cycle; negative solutions in all leaf nodes means that the graph is non-Hamiltonian.

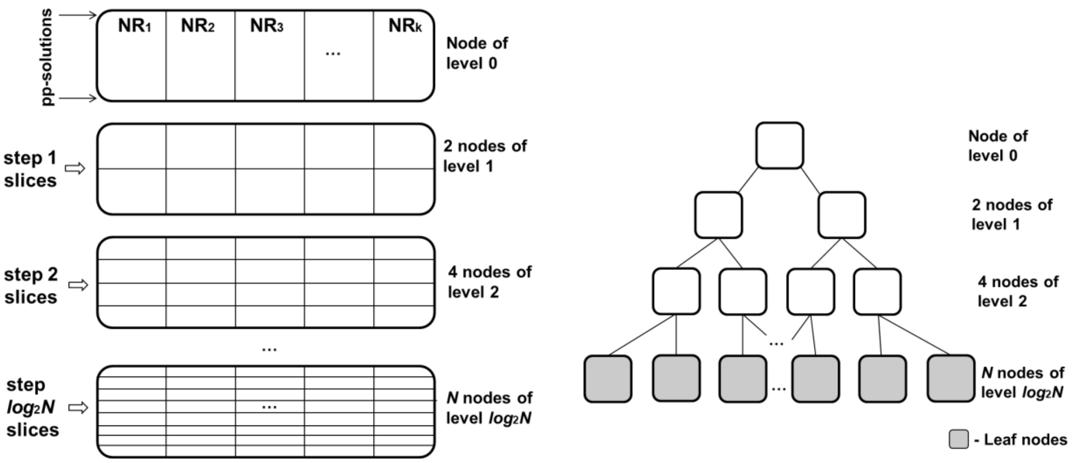


Fig. 15. Two views of the subproblem nodes in scheme 1

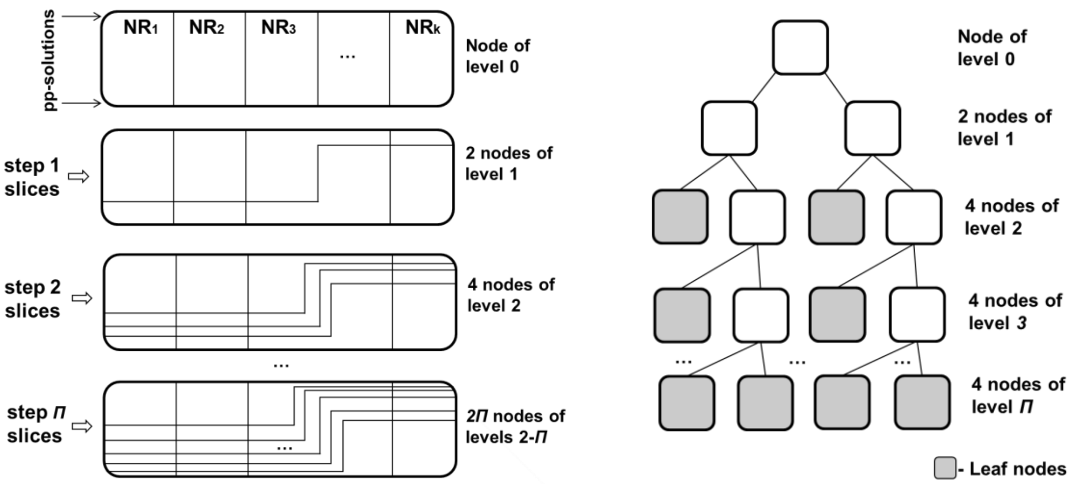


Fig. 16. Two views of the subproblem nodes in scheme 2

Scheme 2 (Fig. 16) assumes that blocking and recognizing pp-solutions are split into two (very unequal) parts in each NR. Then, within the first part, the number of pp-solutions in some NRs can be decreased very slowly and in others very fast. This bias can be stable for some regions. Then, we can assume that the number of splitting steps is of the Π level (Π is the minimum number of blocking or recognizing pp-solutions in NRs at the moment of the process beginning; it is much less than N^2). However, the number of the scheme leaf nodes is not of the 2^{Π} level, but the 2Π level. This is because at each splitting step, the second part operations are employed for extracting invariants in all regions of two nodes of the tree scheme and for solving the corresponding systems of equations. As a result, from four nodes of each level only two are used in splitting for the next level.

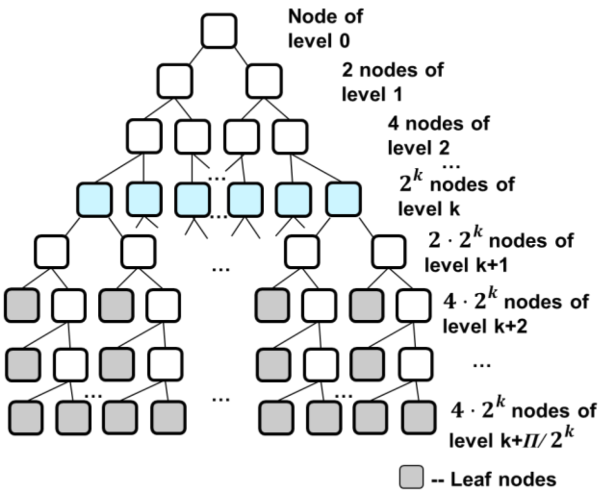


Fig. 17. Subproblem nodes in a combination of schemes 1 and 2

Scheme 3 (Fig. 17) is a combination of schemes 1 and 2. First, it is assumed that blocking and recognizing pp-solutions are split into two (more or less equal) parts for $\boldsymbol{k}$ steps ($\boldsymbol{k} < \boldsymbol{log_2N}$). Then, in each scheme node of step $\boldsymbol{k}$, scheme 2 is implemented. The number of these scheme nodes is evaluated by $2^{\boldsymbol{k}}$ and the minimum number ($\boldsymbol{\Pi'}$) of blocking and recognizing pp-solutions in NRs by $\boldsymbol{\Pi/2^k}$ ($\boldsymbol{\Pi}$ is the minimum number of blocking and recognizing pp-solutions at the moment of the scheme 1 beginning). In other words, within the first part, the total number of the scheme leaf nodes created after $\boldsymbol{\Pi'}$ steps of scheme 2 is of the $2\times2^{\boldsymbol{k}}\boldsymbol{\Pi}/2^{\boldsymbol{k}} = 2\boldsymbol{\Pi}$ level and, within the second part, the number of nodes, where extracting the invariants and solving the systems of equations are possible, is also of the $2\boldsymbol{\Pi}$ level.

Scheme 4 is similar to scheme 2; it assumes that blocking and recognizing pp-solutions are split into two (unequal) parts in each NR, but the part inequality is not extremal; for example, in proportion 1 to 3 (or 1 to 4). Then, within the first part (under conditions of the stable bias of a worse case for us), the fastest decrease of the pp-solutions in corresponding regions requires $\boldsymbol{p}$ steps to reach factor $\boldsymbol{N}$ with $4^{\boldsymbol{p}} = \boldsymbol{N}$ and $\boldsymbol{p} = (log_2\boldsymbol{N})/2$ and the slowest decrease requires $\boldsymbol{m}$ steps to reach factor $\boldsymbol{N}$ with $(4/3)^{\boldsymbol{m}} = \boldsymbol{N}$ and $\boldsymbol{m} = (log_2\boldsymbol{N})/\boldsymbol{l}og_2(4/3)$. As a result, within the second part, the number of the scheme leaf nodes involved can be estimated by $O(\boldsymbol{N^{1/2}}log_2\boldsymbol{N})$; this is because $\boldsymbol{N^{1/2}}$ is the number of the scheme nodes at level $\boldsymbol{p}$, $log_2\boldsymbol{N}(1/\log_2(4/3) - ½)$ is the number of the levels after level $\boldsymbol{p}$, $2\boldsymbol{N^{1/2}}$ is the number of chains going down after level $\boldsymbol{p}$ and $(log_2\boldsymbol{N}(1/\log_2(4/3) - ½))/2$ is an average number of leaf nodes in chains mentioned above.

Any other combinations of these basic schemes and schemes with systematic or chaotic changes of biases in splitting the pp-solutions align (in some sense) pp-solution slices and also lead to polynomial numbers of the leaf nodes.

6. About correctness of the approach

Our approach is based on the existence of redundancy in the graph regions, on Hamiltonicity-preserving eliminations of the redundancy and on discovering regional invariants within the process of the eliminations and the pp-solution splits across the regions. This allows efficient reducing of the Hamiltonicity problem to a polynomial number of the subproblems, which are related to different slices of pp-solutions and solved as the systems of linear algebraic equations. The following points are behind the approach.

Statement 1: Redundancy eliminations in regions preserve at least one Hamiltonian cycle in the graph, if such cycles exist. Data dependence on decisions in neighbouring regions excludes introducing incompatibility, which is not derived from the graph features.

Statement 2: An **r**-invariant extracted from a set of pp-solutions in a region is valid on any subset of these pp-solutions. Removing additional pp-solutions from the set preserves the **r**-invariant extracted.

Statement 3: An **r**-invariant discovered in a region is valid for pp-solutions of any region including edges of the **r**-invariant.

Statement 4: Removing blocking pp-solutions in any region induces removing blocking pp-solutions in a chain of regions leading to a root region from where recognizing an invariant pattern has been initiated. This allows lossless consideration of the subproblems, based not on decomposing the graph, but on splitting the sets of pp-solutions across the regions.

Statement 5: The non-Hamiltonicity is discovered by recognizing failures for the complementing invariant patterns in a region or by inappropriate solutions of the systems of equations.

All steps of the approach have polynomial complexity including memory we should expend for saving pp-solutions in all possible sets of overlapping regions; a rough layout of estimations gives us the following:

- The number of basic regions < ***N***: we need ***N***/2 **r**-invariants to compose a system of ***M*** equations (for graph ***G*** ***M***=3***N***/2); the **r**-invariants should be from all regions of the graph; let the average number of nodes in such a region be Δ, then ***N***/Δ regions can cover graph ***G*** without overlapping; overlapping with four neighbouring regions (as we applied) can involve two additional layers of the graph tiling to get 3***N***/Δ regions; in any case, 3 < Δ,

- The number of neighbouring regions in tree chains (TCs) for one basic regions < $\boldsymbol{N}$: this is because of the previous item; the total number of the regions covering graph $\boldsymbol{G}$ is less than $\boldsymbol{N}$,

- The number of pp-solutions in a region is O($\boldsymbol{N}^2$): the number of different B-set values is estimated by 2^δ, where δ is the number of the region B-edges; we consider regions with $\sigma_0 log_2\boldsymbol{N}$ ($\sigma_0 \leq 1$) B-edges, this means that 2^δ is O($\boldsymbol{N}$); the number of different pp-solutions for each feasible B-set value is estimated by 2^ω, where ω+1 is the number of region interior faces; we consider regions with RR-index = ©+1 (0≤©≤©, © is a constant); this means that ω = ©+$log_2\boldsymbol{N}$ and $2^\omega = 2^{©}\boldsymbol{N}$ = O($\boldsymbol{N}$); in fact, an essential number of expected pp-solutions is involved in internal cycles of **H**-valued edges or incompatible with **n**-invariants. In any case, the product of the number of different B-set values and the number of different pp-solutions for each feasible B-set value gives us O($\boldsymbol{N}^2$).

- The time to obtain these pp-solutions is O($\boldsymbol{N}^2(log_2\boldsymbol{N})^3$); O($(log_2\boldsymbol{N})^3$) is the time for solving the regional system of equations for obtaining a pp-solution,

- Total number of pp-solutions in a basic region and in all neighboring regions related to TCs is O($\boldsymbol{N}^3$), the total number of pp-solutions in all basic regions and in all related NRs is O($\boldsymbol{N}^4$) (in fact, arranging the region reuse can allow the corresponding pp-solutions' reuse, too; then $\boldsymbol{N}^4$ can be replaced by $\boldsymbol{N}^3$),

- The number of operations for selecting the blocking edge pairs in all NRs for all basic regions (under the condition that pp-solutions in the TCs' regions obtained) is O($\boldsymbol{N}^6(log_2\boldsymbol{N})^2$): in each region involved, the number of edges is of $log_2\boldsymbol{N}$ level and the number of 2**F** pairs is O($(log_2\boldsymbol{N})^2$) ; the number of comparisons in pp-solutions of one NR for all pairs is O($\boldsymbol{N}^2(log_2\boldsymbol{N})^2$) and for $\boldsymbol{N}$ NRs and $\boldsymbol{N}$ basic regions is O($\boldsymbol{N}^4(log_2\boldsymbol{N})^2$). $\boldsymbol{K}$ splitting steps for $\boldsymbol{N}/2$ invariant patterns leads to O($\boldsymbol{K}\times\boldsymbol{N}^5(log_2\boldsymbol{N})^2$) ($\boldsymbol{K} < \boldsymbol{N}^2$).

7. Conclusion

A regional invariant method based on redundancy eliminations has been presented and the polynomial solution of the graph Hamiltonicity problem has been demonstrated. The method includes searching basic and neighboring regions, considering all possible partial solutions (pp-solutions) within such regions, and removing redundant pp-solutions. Based on the concept of boundary communicators, a feasible set of the pp-solutions is arranged for each boundary case of the region. The redundancy elimination preserves only one representative pp-solution from each of such sets. Introducing **H** and **F** symbolic parameters allows applying systems of linear algebraic equations for the incompatibility and redundancy eliminations, extracting regional invariants from the representative pp-solutions, as well as for composing a system of the linear equations and solving a Hamiltonicity problem as a whole. Within the method, the invariant pattern recognition based on arranging the region chains for 2**F** pair eliminations of a remote impact is also applied. 2**F** pair eliminations employ a large-scale type of redundancy which is related not to decomposing the graph, but to splitting the sets of pp-solutions across the regions. This allows efficient reduction of the Hamiltonicity problem to a polynomial number of the subproblems, which are solved as the systems of linear algebraic equations.

Existence of redundancy and how to eliminate it have been demonstrated. In fact, this is an answer to questions about where the complexity disappears to and why P=NP. Different NP-complete problems are related to different types of redundancy. Therefore, solving such problems should commence with discovering corresponding types of redundancy and related equations with symbolic parameters, as well as with understanding usefulness of possibly failed retrieval operations.

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Appendix 1

A subset of rules for reasoning about edge values	
$E_j = \mathbf{F} - \mathbf{H} + E_k \longrightarrow E_j = \mathbf{F}, E_k = \mathbf{H}$	$E_j = \mathbf{H} - \mathbf{F} + E_k \longrightarrow E_j = \mathbf{H}, E_k = \mathbf{F}$
$E_j = 2\mathbf{H} - E_k \longrightarrow E_j = \mathbf{H}, E_k = \mathbf{H}$	$E_j = 2\mathbf{F} - E_k \longrightarrow E_j = \mathbf{F}, E_k = \mathbf{F}$
$E_j = 3\mathbf{H} - E_k - E_n \longrightarrow E_j=E_k=E_n= \mathbf{H}$	$E_j = 2\mathbf{H} - 3E_k + 2E_n \longrightarrow E_j = E_k = E_n = \mathbf{H}$
$E_j = \mathbf{H}–2\mathbf{F} + E_k + E_n \longrightarrow E_n=E_k= \mathbf{F},\ E_j = \mathbf{H}$	$E_j = 2\mathbf{H}-\mathbf{F} + E_n - E_k \longrightarrow E_j=E_k=\mathbf{H}, E_n = \mathbf{F}$
$E_j = 2E_k – E_n \longrightarrow E_k = E_n$	$E_j = 2\mathbf{H} + \mathbf{F} - 2E_k \longrightarrow E_j = \mathbf{F}, E_k = \mathbf{H}$
$E_j = (\mathbf{F} + E_k)/2 \longrightarrow E_j = E_k= \mathbf{F}$	$E_j = (2\mathbf{H} + \mathbf{F} - E_k)/2 \longrightarrow E_j = \mathbf{H}, E_k = \mathbf{F}$
$E_j = (E_k + E_n)/2 \longrightarrow E_k = E_n$	$E_j = (E_k + 3E_n –2\mathbf{F})/2 \longrightarrow E_j = E_k = E_n =\mathbf{F}$
$E_j = (4\mathbf{H} + \mathbf{F} - 3E_k)/2 \longrightarrow$ **invalid**	$E_j = (2E_k + 2E_n – \mathbf{F})/3 \longrightarrow E_j=E_k=E_n = \mathbf{F}$
$E_j = (E_k+E_n + \mathbf{H})/3 \longrightarrow E_j=E_k=E_n=\mathbf{H}$	$E_j = z\mathbf{H} - E_1-E_2 -\ldots- E_{z-1} \longrightarrow E_j=E_1=\ldots=E_{z-1}=\mathbf{H}$
$(E_j = 2\mathbf{H}+\mathbf{F} - E_n - E_k)$ & $(E_z = E_n + E_k - \mathbf{F}) \longrightarrow E_j=E_z=\mathbf{H}, E_n + E_k = \mathbf{F} + \mathbf{H}$	$(E_j = E_n + E_k – \mathbf{H})$ & $(E_n + E_k = \mathbf{H}+\mathbf{F}) \longrightarrow E_j = \mathbf{F}$
$(E_k + E_n = \mathbf{F} + \mathbf{H})$ & $(E_k + E_j = \mathbf{F} + \mathbf{H}) \longrightarrow E_n = E_j$	$(E_j = 2\mathbf{H} + \mathbf{F} - E_n - E_k)$ & $(E_n = E_k) \longrightarrow E_n = E_k = \mathbf{H}, E_j = \mathbf{F}$
$E_j = 3\mathbf{H} + \mathbf{F} – E_k - 2E_n \longrightarrow E_n = \mathbf{H}$	$(E_j = 3\mathbf{H} + \mathbf{F} – E_k - 2E_n)$ & $(E_z = E_k+3E_n - 2\mathbf{H} - \mathbf{F}) \longrightarrow E_n = \mathbf{H},\ E_k = \mathbf{F}$
$E_j = \sum_{i=1}^{k} c_i H_i - \sum_{t=1}^{p} d_t H_t \longrightarrow \sum_{i=1}^{k} c_i - \sum_{t=1}^{p} d_t = 1$ (c_i, d_t – integers; H_i, H_t – edges or **H** or **F**)	$E_j = E_k - E_n + \mathbf{H} + \mathbf{F} \longrightarrow$ **invalid**

Appendix 2

Maximal possible number of irremovable pp-solutions for k pairs of H-valued B-edges

k=1: $s_1 = 1$
k=2: $s_2 = 3s_1 – 1 = 3 – 1$
k=3: $s_3 = 3(3 – 1) – 1 = 3^2 – 3 - 1$
k=4: $s_4 = 3(3^2 – 3 -1) – 1 = 3^3 - 3^2 – 3 - 1$
k=5: $s_5 = 3(3^3 - 3^2 – 3 -1) – 1 = 3^4 - 3^3 - 3^2 – 3 - 1$
…
k=p: $s_p = 3(3^{p-2} – 3^{p-3} – …– 3 -1) – 1 = 3^{p-1} – 3^{p-2} –… – 3^2 – 3 – 1 =$ $\mathbf{3^{p-1}/2 +1/2}$

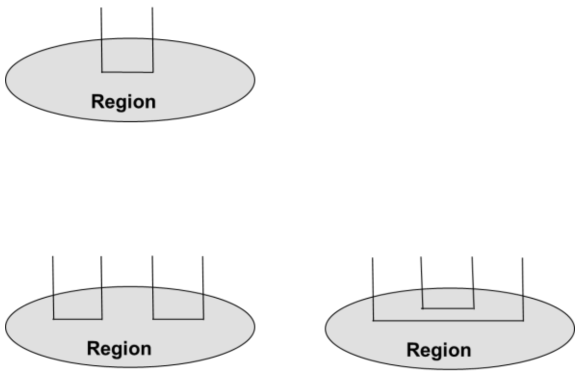

Fig. A. Schemes of irremovable pp-solutions for 1 and 2 pairs of H-valued B-edges

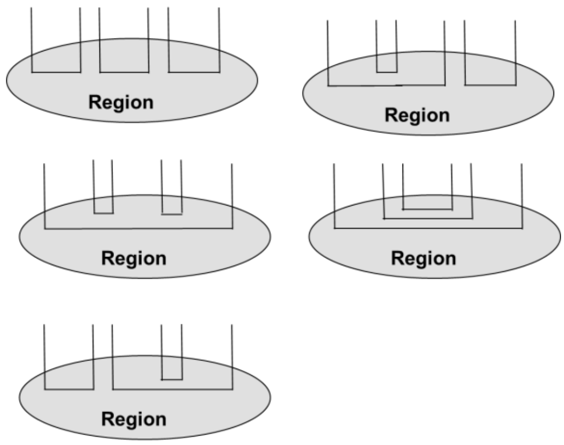

Fig. B. Schemes of irremovable pp-solutions for 3 pairs of H-valued B-edges

The number of irremovable pp-solutions for a B-set value with 2**k** **H**-valued B-edges (1≤**k**≤**p**) and 2**p**+1 faces is limited by $3^{k-1}/2+1/2$ and the total number of (removable and irremovable) pp-solutions is limited by 2^{2p} (**k** and **p** are of a logarithmic level). Inequality $2^{2p} > 3^{k-1}/2+1/2$ is enhanced for regions with greater values of RR-index.

Appendix 3

2F pair eliminations in the NR chains

1. In each pp-solution intended for the elimination (that is a pp-solution blocking an invariant recognition in a BR (basic region)), we find all pairs (E_j,E_k) of edges (from overlapping area of NR-BR) with E_j=E_k=**F** values; the pairs of all pp-solutions are united in one set (let their number be φ).
2. For each pair (E_p,E_z) of the set, we check all clusters of pp-solutions in the NR for possible REs' removal of the BR pp-solutions that block the invariant recognition. Within a cluster, REs are fully performed only if there are not decision alternatives. Otherwise, eliminations with alternative pp-solutions (where E_p=E_z=**F** is invalid) are postponed (for flexibility in further operations). As a result, for each pair we find a number (which can be 0) of blocking pp-solutions with E_p=E_z=**F**. The pairs are sorted by ascending value of the number: P_1, P_2, P_3, …, $P_φ$,
3. A subset of pairs, with the zero sum of the numbers and related to all blocking pp-solutions in the RUC, leads to recognizing the invariant pattern in RUC, under conditions of independency of REs induced by different pairs.
4. In cases with dependencies, for each pair P_j (1 ≤ j ≤ φ) from step 2, sequences of the following type are created:

   P_1, P_2, P_3, …, $P_φ$
   P_2, P_1, P_3, …, $P_φ$
   …
   P_j, P_1, P_2, …, P_{j-1}, P_{j+1}, …, $P_φ$
   …
   $P_φ$, P_1, P_2, …, $P_{φ-1}$.

   Then, REs on the NR are performed according to each sequence and the number of blocking pp-solutions remaining is extracted. After that, a sequence with the minimum of the blocking pp-solutions is selected for further operations in a new NR.
5. In the new NR, steps 1, 2, 3, and 4 are performed in areas that are arranged by replacing the BR by the NR and the NR by a new NR. A remaining pp-solution is recognized as such, if it is not involved in local cycles (within the compound region including BR and chain's NRs).
6. This step is repeated for new regions until recognizing the invariant pattern or covering the graph by the tree construction of the region chains.

2**F** pair eliminations in the NR chains are Hamiltonicity-preserving eliminations that support arranging remote relations of blocking/unblocking types between pp-solutions. Blocking relations provide a set of pp-solutions in G-region with a conditional impact on a set of pp-solutions in D-region. The conditional impact means that removing (because of some reason) the set from G-region leads via an NR chain to removing the corresponding set from D-region. Unblocking relations provide a set of the D-region pp-solutions that are a basis for discovering, via an NR chain, the G-region pp-solutions, which unconditionally (because of redundancy eliminations) remove the D-region pp-solutions.

Appendix 4

Involvement of the blocking NR pp-solutions in internal H-valued circles

Here, we clarify checking possible involvement of the blocking NR pp-solutions in internal **H**-valued circles of a compound (c-) region including the RUC and NRs. Such involvement eliminates blocking pp-solutions related and avoids additional operations for taking them into account. The checking is performed at each step of the NR chain construction.

In the first step from the blocking RUC pp-solutions, we extract B-communicator pairs of B-edges that are internal to NR_1(b) (see Fig. 14 (a)). Then based on these pairs for each blocking NR_1 pp-solution, we check its involvement in the cycles within the c-region of NR_1-RUC. In addition, B-communicator pairs (being internal in NR_2(b)) of the c-region NR_1-RUC pp-solutions are saved (if any).

In the second step, based on the B-communicator pairs saved, we check the involvement of blocking NR_2 pp-solutions in the circles within c-region NR_2-NR_1-RUC and preserve its B-communicator pairs (being internal in NR_3(b)).

Other steps are similar to the previous ones. In fact, instead of a pp-solution of $NR_{\mathbf{k}}$, its part (a pp-solution of NR_k(t)) is employed. The B-communicator pairs saved at step **k** represent returning paths within c-region $NR_{\mathbf{k}}$-…-NR_2-NR_1-RUC.

H-valued circles can be discovered not only in the linear chains, but also in the NR chain loop (like at Fig. 14 (b)). For reaching this goal, in addition to the returning paths, we save B-communicator paths, each of which has an internal for NR_{j+1}(b) B-edge of c-region $NR_{\mathbf{j}}$-NR_{j-1}-…-NR_1-RUC and an internal for NR_k(r) B-edge of the same c-region. With such data saved, we can efficiently discover the internal cycles inside some c-regions.